Modified Compact Central Finite Difference Schemes For The Simulations Of Wave Equation At Any Wave Number

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Abstract

In this paper, we present modified central finite difference (C.F.D.) schemes for solution of the wave equation. The modified schemes (a) provide highly accurate solutions at nodes of the spatial grid for all time steps; (b) preserve the compact stencil structure as of standard C.F.D. scheme and higher order accuracy is achieved without implementation of new code; (c) offer highly accurate solutions for low as well as high wave numbers without use of fine grid. Finally, in order to display superiority of modified C.F.D. schemes numerical computations and graphs are presented for applications such vibrating string and wave propagations compared with standard C.F.D. schemes.

keyword

Compact Finite Difference Schemes, Numerical dispersion, Numerical dissipation, Wave equation, High wave number, wave propagations, Vibrational analysis

المخططات المدمجة للفروق المنتهية المركزية المعدلة لمحاكاة معادلة الموجة في أي رقم موجة

في هذه الورقة، نقدم مخططات معدلة للفروق المنتهية المركزية (C.F.D) لحلّ معادلة الموجة. توفر المخططات المعدلة (أ) عقدًا دقيقًة للغاية لحلّ الشبكة المكانية لجميع المراحل الزمنية؛ (ب) الحفاظ على هياكل الاستنسل المدمجة لمعيار C.F.D يتم تحقيق مخطط ودقة ترتيب عالية دون تطبيق رمز جديد؛ (ج) تقديم حلول دقيقة للغاية لأعداد الموجات المنخفضة وكذلك العالية دون استخدام الشبكة الدقيقة. أخيرًا، من أجل إظهار تفوق C.F.D تعديل الرسوم البيانية والحسابات الرقمية يتم تقديمها للتطبيقات مثل انتشار الأحبال والموجات الاهتزازية مقارنة مع معيار مخططات C.F.D.

1 Introduction

In todays era of science and technology, we are all surrounded by waves in the form of microwave oven, cell phones, ultrasound scans and radar systems etc. Sound basics of waves and understanding of their propagating nature can lead to improve life of human existance and can help us for future explorations of both nature and universe around us. However, complex mathematical form of such phenomenon either time independent or dependent poses a challenge to physicists, engineers and mathematicians to find explicit analytical solutions. This is often tedious and in many circumstances even not realistically possible. This invites numerical analysts to step in, and make efforts to make an impossible thing a possible one. Hence, efficient and reliable numerical methods are required to deal with such problems. In history a tremendous work has come forth by many [1, 2, 3, 4, 5, 6, 7, 8]. Unfortunately, by use of these discretization schemes issues known as numerical dispersion and numerical dissipation in wave propagations literature are born .

We now list a few of remarkable contributions regarding time independent form of wave equation for large wave numbers. In [9, 10, 11] compact finite difference schemes were presented for Helmholtz equation. Properties such as convergence and accuracy were discussed in detail. However, Nehrbass, Jevtic and Lee [12] in 1998, introduced a novel representation of the second order central finite difference (C.F.D.) scheme by redefining the usual second order C.F.D. approximation with a central node of $2\cos(\kappa h)+(\kappa h)^2$ instead of 2. This idea was further extended by Yau and Li [13] where they presented exact construction of non-reflecting boundary condition. In [14], an alternative approach was adopted for the construction of modified central finite difference schemes for simulations of Helmholtz equation at any wave number for uniform grids only using *Bloch wave property*. The idea proposed in [14] was further extended for the construction of modified C.F.D. schemes for adaptive grids [15]. In this work we adopt the approach presented in [14, 15] for time harmonic wave equation and extend this for time dependent wave equation.

The organization of the paper is as follows. In Section 2, we present framework for the construction of modified compact central finite difference schemes. In Section 3, numerical examples are presented and in final section schemes are constructed for two dimensional problem and dispersion analysis is presented.

2 Framework for the construction of modified compact central finite difference schemes

We consider the one dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad u=u(x,t) \text{ and } c>0 \text{ represents wave speed} \tag{1}$$

in order to motivate the ideas for the construction of modified compact C.F.D. schemes.

2.1 Construction of modified compact explicit C.F.D. scheme

For the construction of exact C.F.D. schemes, we adopt the idea of Nehrbass and Li presented in [12] for Helmholtz equation. Here we use this idea for wave equation (1). We replace partial derivatives, $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial t^2$, present in (1), by standard second order C.F.D. approximation such that coefficient of middle node 2 is replaced with α giving

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{1}{h^2}\left[u_{m+1}^n - \alpha u_m^n + u_{m-1}^n\right] + O(h^2), \\ \text{and} \qquad \frac{\partial^2 u}{\partial t^2} &= \frac{1}{\ell^2}\left[u_m^{n+1} - \alpha u_m^n + u_m^{n-1}\right] + O(\ell^2). \end{aligned}$$

Inserting above approximations in (1) and performing ordinary simplifications, we get

$$u_m^{n+1} = (cr)^2[u_{m+1}^n + u_{m-1}^n] + \alpha(1-(cr)^2)u_m^n - u_m^{n-1} + O(h^2) + O(\ell^2) \tag{2}$$

with $r = \ell/h$. The unknown α in (2) was termed as dispersion reducing parameter in [12]. A solution of the form $u_m^n = e^{i(m\kappa h - n\omega\ell)}$, known as plane wave solution when substituted in (2) gives after simplifications

$$[2\left((cr)^2\cos(\kappa h) - \cos(\omega\ell)\right) + \alpha(1-(cr)^2)]u_m^n = 0$$

where $\kappa, \omega > 0$, are the wave number and frequency respectively. Since $u_m^n \neq 0$, and solving for α, we have

$$\alpha = \frac{2[(cr)^2\cos(\kappa h) - \cos(\omega\ell)]}{((cr)^2 - 1)}. \tag{3}$$

With this value of α, (2) results into compact explicit C.F.D. scheme given by

$$u_m^{n+1} = (cr)^2(u_{m-1}^n + u_{m+1}^n) + 2[\cos(\omega\ell) - (cr)^2\cos(\kappa h)]u_m^n - u_m^{n-1}. \tag{4}$$

Interestingly, (4) can also be constructed using a very powerful tool known as *Bloch wave property* [14] defined by

$$u_{m+j}^n = u_m^n e^{ij\kappa h} \quad \text{and} \quad u_m^{n+\tilde{j}} = u_m^n e^{i\tilde{j}\omega\ell} \quad \forall j, \tilde{j} \in \mathbb{Z} \text{ and } \ell, h > 0 \tag{5}$$

therefore, we have

$$u_{m-1}^n + u_{m+1}^n = 2\cos(\kappa h)u_m^n \quad \text{and} \quad u_m^{n+1} + u_m^{n-1} = 2\cos(\omega\ell)u_m^n. \tag{6}$$

Using (6) in standard explicit C.F.D. scheme with 2 as the coefficient of the middle node, we have

$$\begin{aligned} &u_m^{n+1} - (cr)^2[u_{m-1}^n + u_{m+1}^n] - 2(1-(cr)^2)u_m^n + u_m^{n-1} \\ &\quad = -2[(1-(cr)^2) - \cos(\omega\ell) - (cr)^2\cos(\kappa h)]u_m^n \end{aligned}$$

Rewriting above results into (4) and it is interesting to note that the use of the Bloch wave property provide exactly the same scheme. Moreover, as given in [14] that with the use of Bloch wave property, one can make schemes of any order exact either forward, backward or central for Helmholtz equation. Considering this modification of standard implicit C.F.D. scheme is presented in next section.

2.2 Construction of modified compact implicit C.F.D. scheme

This section is devoted for the construction of modified implicit scheme of (1) by replacing time derivative $\partial^2 u/\partial t^2$ with standard second order C.F.D. approximation whereas spatial derivative $\partial^2 u/\partial x^2$ is replaced with weighted average, such that middle node coefficient 2 is replaced with α given by

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\ell^2}\left[u_m^{n-1} + u_m^{n+1} - 2u_m^n\right] + O(\ell^2) \quad \text{and}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} = \frac{1}{4h^2}\Bigg\{ & u_{m+1}^{n+1} - \alpha u_m^{n+1} + u_{m-1}^{n+1} + 2\left(u_{m-1}^n - \alpha u_m^n + u_{m+1}^n\right) \\ & + u_{m+1}^{n-1} - \alpha u_m^{n-1} + u_{m-1}^{n-1}\Bigg\} + O(h^2). \end{aligned}$$

Substitution of above approximations in (1) after simplifications give

$$\begin{aligned} \frac{(cr)^2}{4}\Bigg\{ & u_{m+1}^{n+1} - \alpha u_m^{n+1} + u_{m-1}^{n+1} + 2\left(u_{m-1}^n - \alpha u_m^n + u_{m+1}^n\right) \\ & + u_{m+1}^{n-1} - \alpha u_m^{n-1} + u_{m-1}^{n-1}\Bigg\} = u_m^{n+1} - 2u_m^n + u_m^{n-1}. \end{aligned} \tag{7}$$

Now, using Bloch wave property given in (5) and performing straight forward calculations, equation (7) gives following value of α:

$$\alpha = 2\cos(\kappa h) + \frac{4}{(cr)^2}\frac{1-\cos(\omega\ell)}{1+\cos(\omega\ell)}. \tag{8}$$

Inserting value of α from (8) into (7) gives

$$-\frac{(cr)^2}{4}\left[u_{m-1}^{n+1}+u_{m+1}^{n+1}\right]=\frac{(cr)^2}{2}\left[u_{m-1}^{n}+u_{m+1}^{n}\right]+\frac{(cr)^2}{4}\left[u_{m+1}^{n-1}+u_{m-1}^{n-1}\right]$$
$$-\left[1+\frac{(cr)^2}{2}\cos(\kappa h)-\frac{\cos(\omega\ell)-1}{\cos(\omega\ell)+1}\right]u_m^{n+1}+\left[2-(cr)^2\cos(\kappa h)-2\frac{\cos(\omega\ell)-1}{\cos(\omega\ell)+1}\right]u_m^{n}$$
$$-\left[1+\frac{(cr)^2}{2}\cos(\kappa h)-\frac{\cos(\omega\ell)-1}{\cos(\omega\ell)+1}\right]u_m^{n-1} \tag{9}$$

which is the required form of modified compact implicit C.F.D. scheme.

2.3 Combined form of modified compact explicit and implicit C.F.D. schemes

We now present combined form of both modified explicit and implicit schemes obtained in (4) and (9) by introducing parameter β given by

$$\left\{1+\beta\left(2(cr)^2\cos(\kappa h)+4\left[\frac{1-\cos(\omega\ell)}{1+\cos(\omega\ell)}\right]\right)\right\}u_m^{n+1}-(cr)^2\beta\left(u_{m-1}^{n+1}+u_{m+1}^{n+1}\right)+$$
$$=(cr)^2(1-2\beta)u_{m-1}^{n}+\left\{8\beta+2(3\beta-1)(cr)^2\cos(\kappa h)+2(1-4\beta)\cos(\omega\ell)\right.$$
$$\left.+8\beta\left[\frac{\cos(\omega\ell)-1}{\cos(\omega\ell)+1}\right]\right\}u_m^{n}-(cr)^2(-1+2\beta)u_{m+1}^{n}+(cr)^2\beta(u_{m-1}^{n-1}+u_{m+1}^{n-1})$$
$$-\left\{1+\beta\left(2(cr)^2\cos(\kappa h)+4\left[\frac{1-\cos(\omega\ell)}{1+\cos(\omega\ell)}\right]\right)\right\}u_m^{n-1}. \tag{10}$$

Choosing $\beta=0$ in (10), results into (4) whilst for $\beta=1/4$, (10) reduces to (9).

Interestingly, series expansions of $\cos(\omega\ell)$ and $\cos(\kappa h)$ present in (10) in terms of $\omega\ell$ and κh results into combined representation of standard explicit and implicit schemes obtained in [16] when $\omega\ell\to 0$ and $\kappa h\to 0$ given below

$$-(cr)^2\beta(u_{m+1}^{n+1}+u_{m-1}^{n+1})+(1+2(cr)^2\beta)u_m^{n+1}=2(1+(cr)^2(2\beta-1))u_m^{n}$$
$$+(cr)^2(1-2\beta)[u_{m-1}^{n}+u_{m+1}^{n}]+(cr)^2\beta[u_{m+1}^{n-1}+u_{m-1}^{n-1}]-(1+2(cr)^2\beta)u_m^{n-1}. \tag{11}$$

This is an attractive feature of modified scheme and is consistent with the requirement of not using finer mesh size especially when simulations are required for high wave numbers.

2.4 Numerical Dispersion of Modified and Standard Schemes: A Comparison

Now to present dispersion analysis of modified compact explicit and implicit C.F.D. schemes, we substitute a plane wave solution of the form $u_m^n = e^{i(m\tilde{\kappa}h - \omega n\ell)}$ in combined form (10) and obtain after straight forward manipulations

$$(cr)^2[\cos(\tilde{\kappa}h) - \cos(\kappa h)]u_m^n = 0$$

where $\tilde{\kappa}$ is known as the discrete wave number. For a non-trivial solution $u_m^n \neq 0$ above equation implies

$$\tilde{\kappa} = \kappa.$$

Therefore the modified schemes (10), do not suffer from issues such as numerical dispersion and dissipation contrary to standard schemes (11) as reported in [14, 16].

3 Numerical Examples

We now test performance of modified schemes by solving problems such as a vibrating string and propagation of wave from left to right in following sections.

3.1 A vibrating string problem

Consider one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \ x \in (0,1) \text{ and } t > 0 \tag{12}$$

with boundary conditions,

$$u(0,t) = u(1,t) = 0 \qquad t > 0 \tag{13}$$

and initial conditions

$$u(x,0) = \sin(\pi x) \ \text{ and } \ \frac{\partial u(x,0)}{\partial t} = u_t(x,0) = 0 \qquad x \in [0,1]. \tag{14}$$

This problem has an exact solution [16] given by

$$u(x,t) = \sin(\pi x)\cos(\pi t). \tag{15}$$

This problem describes a string tied at both ends vibrating up and down with $\sin(\pi x)$ as initial displacement and zero initial velocity. Initial and boundary conditions are chosen such that continuity between them is preserved with $c^2 = 1$ as speed of the medium. Spatial $(0,1)$ and time $t \geq 0$ domains are discretized into N subintervals of length $h = 1/N$ and length ℓ respectively. This means each node is represented by coordinates of the form: $(x_m, t_n) = (mh, n\ell)$ for $m = 0, 1, 2, \cdots, N$ and $n = 0, 1, 2, \cdots$. Moreover, for this problem error is defined by $Z_m^n = |u_m^n - \tilde{U}_m^n|$ [16] with u_m^n and $\tilde{U}_m^n$ as exact and numerical solutions respectively. For time $t = 1$ highest value of the error is picked for spatial range $x \in (0,1)$.

3.2 Combined form of standard and modified C.F.D. schemes

First of all, we give combined form of standard and modified C.F.D. schemes given for $c = 1$

$$\begin{aligned} & -r^2\beta(u_{m+1}^{n+1} + u_{m-1}^{n+1}) + (1 + 2r^2\beta)u_m^{n+1} \\ &= 2(1 + 2r^2\beta - r^2)u_m^n + r^2(1 - 2\beta)[u_{m-1}^n + u_{m+1}^n] \\ &\quad + r^2\beta[u_{m+1}^{n-1} + u_{m-1}^{n-1}] - (1 + 2r^2\beta)u_m^{n-1} \end{aligned} \tag{16}$$

and

$$\begin{aligned} &\left\{1 + \beta\left(2r^2\cos(h\pi) + 4\left[\frac{1 - \cos(\ell\pi)}{1 + \cos(\ell\pi)}\right]\right)\right\}u_m^{n+1} - r^2\beta(u_{m-1}^{n+1} + u_{m+1}^{n+1}) \\ &= r^2(1 - 2\beta)[u_{m-1}^n + u_{m+1}^n] + \Bigg\{8\beta + 2(-1 + 3\beta)r^2\cos(h\pi) \\ &\quad + 2(1 - 4\beta)\cos(\ell\pi) + 8\beta\left(\frac{\cos(\ell\pi) - 1}{\cos(\ell\pi) + 1}\right)\Bigg\}u_m^n \\ &\quad + r^2\beta(u_{m-1}^{n-1} + u_{m+1}^{n-1}) - \left\{1 + \beta\left(2r^2\cos(h\pi) + 4\left[\frac{1 - \cos(\ell\pi)}{1 + \cos(\ell\pi)}\right]\right)\right\}u_m^{n-1}. \end{aligned} \tag{17}$$

Above schemes are valid for all internal nodes of the spatial grid and for boundary nodes solution is already known as Dirichlet boundary conditions are chosen. Also, above schemes result into standard explicit and standard implicit forms for $\beta = 0$ and $\beta = 1/4$ respectively.

3.3 Solution at first time step for standard and modified schemes

We now consider initial velocity given in (14), that is

$$u_t(x, 0) = 0.$$

and replacing time derivative by second order C.F.D. approximation gives

$$\frac{1}{2\ell}\left(u_m^{n+1}-u_m^{n-1}\right)=0\Rightarrow u_m^{n-1}=u_m^{n+1}. \tag{18}$$

Inserting (18) into (16) and (17), we obtain desired schemes for initial time step for standard

$$\begin{aligned}-r^2\beta(u_{m+1}^{n+1}+u_{m-1}^{n+1})+2(1+2r^2\beta)u_m^{n+1}\\ =2(1+2r^2\beta-r^2)u_m^n+r^2(1-2\beta)[u_{m-1}^n+u_{m+1}^n]\\ +r^2\beta[u_{m+1}^{n-1}+u_{m-1}^{n-1}]\end{aligned}$$

and modified scheme

$$\begin{aligned}2\left\{1+\beta\left(2r^2\cos(h\pi)+4\left[\frac{1-\cos(\ell\pi)}{1+\cos(\ell\pi)}\right]\right)\right\}u_m^{n+1}-r^2\beta(u_{m-1}^{n+1}+u_{m+1}^{n+1})\\ =r^2(1-2\beta)[u_{m-1}^n+u_{m+1}^n]+\Bigg\{8\beta+2(-1+3\beta)r^2\cos(h\pi)\\ +2(1-4\beta)\cos(\ell\pi)+8\beta\left(\frac{\cos(\ell\pi)-1}{\cos(\ell\pi)+1}\right)\Bigg\}u_m^n+r^2\beta(u_{m-1}^{n-1}+u_{m+1}^{n-1})\\ -2\sin(\ell\pi)\sin(\pi x)\sin(\pi t)\left\{1+\beta\left(2r^2\cos(h\pi)+4\left[\frac{1-\cos(\ell\pi)}{1+\cos(\ell\pi)}\right]\right)\right\}.\end{aligned}$$

3.4 Analysis of vibrating string problem for all schemes

Table 1: Errors analysis for vibrating string problem with standard and modified explicit schemes

	One Vibrating Mode		**Seven Vibrating Modes**	
Scheme	**Standard**	**Modified**	**Standard**	**Modified**
r=0.25	$0.73*10^{-4}$	$1.99*10^{-15}$	1.6328	$8.88*10^{-16}$
r=0.5	$0.47*10^{-4}$	$2.22*10^{-16}$	1.9286	$6.66*10^{-16}$
r=1	0	0	0	0
r=2	$0.82*10^{-3}$	$1.33*10^{-13}$	67163	$4.49*10^{-12}$
r=5	$0.99*10^{-1}$	$8.43*10^{-14}$	2994.5	$5.16*10^{-14}$

Comparison of results obtained with standard and modified C.F.D explicit schemes in case of single as well as seven vibrating modes are given in Table 1 for constant value of $h=0.1$

Table 2: Errors analysis for vibrating string problem with standard and modified implicit schemes

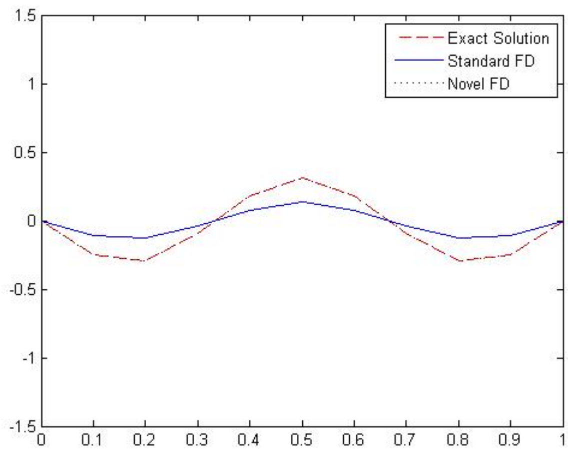
	One Vibrating Mode		Seven Vibrating Modes	
Scheme	**Standard**	**Modified**	**Standard**	**Modified**
r=0.25	$0.10*10^{-3}$	$2.99*10^{-15}$	1.24764	$1.8*10^{-12}$
r=0.5	$0.18*10^{-3}$	$1.33*10^{-15}$	0.49350	$5.42*10^{-12}$
r=1	$0.72*10^{-3}$	0	0.59156	$6.1*10^{-13}$
r=2	$0.59*10^{-2}$	$2.22*10^{-16}$	0.53966	$2.13*10^{-12}$
r=5	0.116021	$2.22*10^{-16}$	1.561386	$9.2*10^{-11}$

and varying values of $r = 0.25, 0.5, 1, 2, 5$, with temporal step sizes $l = 0.025, 0.05, 0.1, 0.2, 0.5$ respectively.

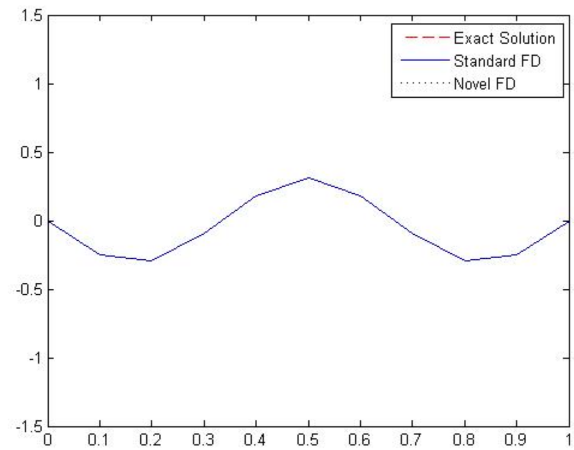
It is evident from Table 1 that highly accurate results are obtained with modified schemes compared with standard schemes even with increasing values of r for either one or seven vibrating modes. Interestingly, standard scheme performed good for lower values of $r = 0.25, 0.5$ and also for $r = 2$ for only one vibrating mode case. However, standard scheme fails to provide accurate results in case of increasing i,e. seven vibrating modes. Therefore, increasing vibrations worsen dispersive and dissipative behaviour in case of standard schemes whereas modified schemes still provides highly accurate results even for large value of $r = 5$. Practical applications require to choose very small values of r in case of standard schemes which adds to prohibitive computational cost to achieve certain level of accuracy. On contrary, modified schemes offers highly promising results even for $r = 5$

For $r = 1$, which is known as magical number [16, 17], both standard and modified explicit schemes provide dispersion less results with zero error which is a proven charactersitic of explicit scheme [16, 17]. Explicit modified scheme constructed also fulfils this characteristic. Standard implicit scheme does not fulfil this characteristic as is evident from Table 2, however modified scheme results into dispersion and dissipation free results for single vibrating mode. Whilst for seven vibrating modes implicit schemes provides highly accurate results. In general modified scheme out performs compared to standard scheme (see Tables 1 and 2).

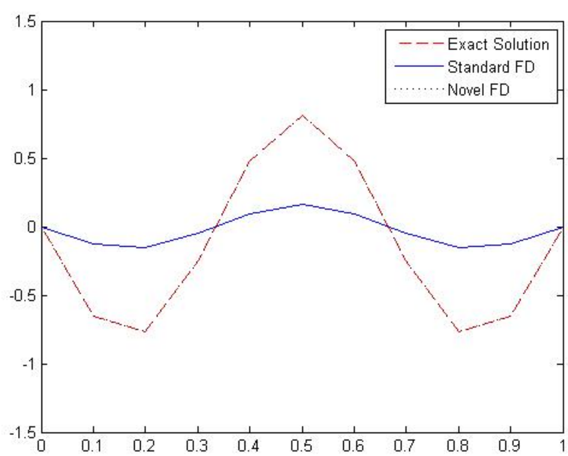
In Figure 1 dispersion error behaviour is shown for three modes of vibrating string using all schemes. Dispersion is prominent for standard explicit and implicit schemes where as modified schemes are perfect interpolate exact solution (see Figure 1 (a),(c) and (d)). Also, dispersion free behaviour is evident in case of magical number i,e. $r = 1$ for both standard and modified explicit schemes (see Figure 1 (b)). However, dispersion is prominent in case of standard implicit scheme in contrast to the modified implicit scheme which is almost dispersion less (see Figure 1 (d)). Same findings were reported in [6, 7] in case of problems with highly oscillatory nature.



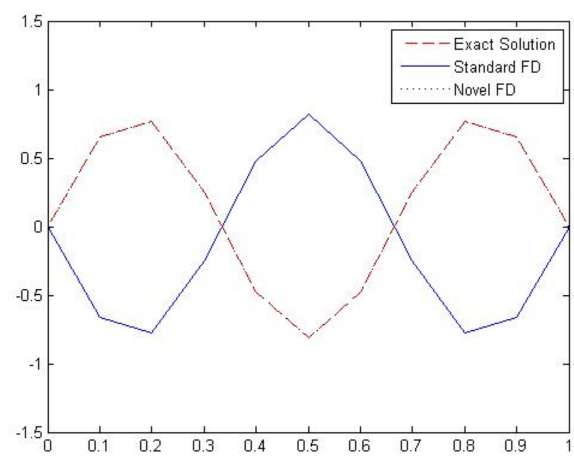
(a) Explicit schemes with $r = 4$



(b) Explicit schemes with $r = 1$



(c) Implicit schemes with $r = 5$



(d) Implicit schemes with $r = 1$

Figure 1: Behavious of all schemes for three vibrating modes.

3.5 A travelling wave from left to right

We now solve (1) with boundary conditions,

$$u(0,t) = e^{i\kappa x} \text{ and } u_x(1,t) + u_t(1,t) = 0 \quad t > 0 \tag{19}$$

and initial conditions

$$u(x,0) = e^{i\kappa x} \quad \text{and} \quad u_t(x,0) = -i\kappa e^{i\kappa x} \quad 0 \leq x \leq 1. \tag{20}$$

The exact solution of this problem is

$$u(x,t) = e^{i\kappa(x-t)}. \tag{21}$$

Above example models a wave travelling from left to right with $-i\kappa e^{i\kappa x}$ initial velocity and medium speed is taken as $c^2 = 1$. Also, same configurations for grid are used as were used in model problem one. For this problem discrete l_∞ norm, defined as [16]

$$E_\infty = max_{m=1,2,\cdots,N} max_{n=1,2,\cdots,} |u_m^n - \tilde{U}_m^n|.$$

is considered. Furthermore, for this problem only standard explicit scheme and its modified version is considered as one can construct modified implicit scheme following similar steps.

3.5.1 Standard explicit C.F.D scheme of second order for travelling wave problem

In order to avoid repetitions, we obtain the desired scheme by choosing $\beta = 0$ into (16)

$$u_m^{n+1} = r^2(u_{m-1}^n + u_{m+1}^n) + 2(1-r^2)u_m^n - u_m^{n-1} \tag{22}$$

with $r = \ell/h$. We can apply scheme (22) only at interior nodes of the grid as using this at right end node i,e. (Nth node) gives fictitious node. Fictitious nodes also known as fantom nodes and always lie outside the boundary. As solution is not known at this node and consequently removal of this node is required which is considered in next section.

3.5.2 Solution at right end node for standard scheme

Consider non-reflecting boundary condition given by

$$u_x(x,t) + u_t(x,t) = 0. \tag{23}$$

Replacing derivatives in (23) by C.F.D. approximation of second order given below

$$u_x = \frac{1}{2h}\left[u_{m-1}^n - u_{m+1}^n\right] + O(h^2) \text{ and } u_t = \frac{1}{2\ell}\left[u_m^{n-1} + u_m^{n+1}\right] + O(\ell^2)$$

gives after simplifications and rearranging

$$u_m^{n+1} = u_{m-1}^n - \frac{u_m^{n-1} - u_m^{n+1}}{r}.$$

Now, inserting above value of u_m^{n+1} in (22), and performing simplifications gives

$$u_m^{n+1} = 2(1-r)u_m^n + \frac{2r^2}{1+r}u_{m-1}^n - \frac{1-r}{1+r}u_m^{n-1}.$$

3.5.3 Solution for first time step for standard scheme

We consider initial velocity given by

$$u_t = -i\kappa u_m^n. \tag{24}$$

Replacing temporal derivative with second order C.F.D. approximation leads to

$$u_m^{n-1} = u_m^{n+1} + 2i\ell\kappa u(x,t). \tag{25}$$

Substituting (25) in (22), we obtain scheme given below to calculate solution value at initial time step

$$u_m^{n+1} = (1 - r^2 - i\ell\kappa)u_m^n + \frac{r^2}{2}\left(u_{m-1}^n + u_{m+1}^n\right).$$

3.5.4 Modified compact explicit C.F.D. scheme for travelling wave problem

For wave propagation problem, the modified compact explicit C.F.D. scheme has following form:

$$u_m^{n+1} = r^2[u_{m-1}^n + u_{m+1}^n] - u_m^{n-1} - 2[r^2\cos(\kappa h) - \cos(\ell\kappa)]u_m^n \tag{26}$$

which can only be used at internal nodes, and therefore for solution at right end node, we have next section.

3.5.5 Solution at right end node for modified scheme

For solution at right end node, we give

$$u_m^{n+1} = u_{m-1}^n + 2i\left(\sin(\kappa h) - \sin(\ell\kappa)\right)u_m^n + u_m^{n-1} - u_m^{n+1}$$

which when inserted into (26) and after simplifications gives

$$u_m^{n+1} = \frac{2}{(1+r^2)}\left[\cos(\ell\kappa) + ir^2\sin(\kappa h) - ir^2\sin(\ell\kappa) - r^2\cos(\kappa h)\right]u_m^n + \left(\frac{2r^2}{1+r^2}\right)u_{m-1}^n - \left(\frac{1-r^2}{1+r^2}\right)u_m^{n-1}.$$

3.5.6 Solution for first time step for modified scheme

Now using exact form of (24) given by

$$u_m^{n+1} = u_m^{n+1} + 2i\sin(\ell\kappa)u_m^n$$

in (26) provides scheme for initial time step

$$u_m^{n+1} = \frac{r^2}{2}[u_{m+1}^n + u_{m-1}^n] - [r^2\cos(\kappa h) + i\sin(\ell\kappa) - \cos(\ell\kappa)]u_m^n.$$

Table 3: Error analysis for wave propagation problem for standard explicit scheme

κ	**(a) r =0.05, h=1**	**(b) r =0.5, h=0.1**	**(c) r =1, h=0.01**
$\mathbf{10^{10}}$	$6.21*10^{10}$	$1.27*10^{10}$	$1*10^{10}$
$\mathbf{10^{8}}$	$7.35*10^{8}$	$2.16*10^{7}$	$9.99*10^{7}$
$\mathbf{10^{6}}$	$9.03*10^{6}$	$1.97*10^{5}$	$9.99*10^{5}$
$\mathbf{10^{4}}$	$3.85*10^{4}$	$9.98*10^{3}$	$9.99*10^{3}$
$\mathbf{10^{2}}$	$6.95*10^{2}$	$2.89*10^{1}$	$1*10^{2}$
$\mathbf{10^{0}}$	1.8158	0.7005	0.9999
$\mathbf{10^{-2}}$	$4.5*10^{-3}$	$7.06*10^{-3}$	$1*10^{-2}$
$\mathbf{10^{-4}}$	$4.52*10^{-5}$	$7.06*10^{-5}$	$9.99*10^{-5}$

3.5.7 Analysis of second problem: A travelling wave with standard and modified explicit schemes

Results obtained for three different combinations $(a)r = 0.05, h = 1$, $(b)r = 0.5, h = 0.1$, and $(c)r = 1, h = 0.01$ with standard and modified explicit schemes are given with varying range of wave numbers κ in Table (3) and Table (4) respectively. It is evident that for high value of wave number such as $\kappa = 10^8, \kappa = 10^9$ or $\kappa = 10^{10}$, modified schemes provide highly accurate results as error stays less than 10^{-5} for all combinations (see Table (4)). However, results obtained using standard scheme are highly erroneous for large values of wave numbers but for very very small values of wave numbers, i,e. when $\kappa h < 1$ reliable results can be seen from Table (3).

Table 4: Errors analysis for wave propagation problem for modified explicit scheme

κ	**(a) r =0.05, h=1**	**(b) r =0.5, h=0.1**	**(c) r =1, h=0.01**
$\mathbf{10^{10}}$	$3.55*10^{-5}$	$6.28*10^{-3}$	$6.66*10^{-5}$
$\mathbf{10^{8}}$	$3.42*10^{-7}$	$2.22*10^{-7}$	$6.51*10^{-7}$
$\mathbf{10^{6}}$	$3.51*10^{-9}$	$1.98*10^{-9}$	$6.67*10^{-9}$
$\mathbf{10^{4}}$	$3.50*10^{-11}$	$2.32*10^{-9}$	$6.71*10^{-11}$
$\mathbf{10^{2}}$	$3.90*10^{-13}$	$1.72*10^{-13}$	$6.49*10^{-13}$
$\mathbf{10^{0}}$	$1.95*10^{-11}$	$2.37*10^{-14}$	$7.72*10^{-15}$
$\mathbf{10^{-2}}$	$9.9*10^{-12}$	$2.62*10^{-14}$	$6.49*10^{-17}$
$\mathbf{10^{-4}}$	$2.02*10^{-11}$	$5.32*10^{-15}$	$0.78*10^{-19}$

For further analysis in Figure 2 results are shown and it is found that wave obtained using standard scheme shows:

- dispersion (phase lag is evident) for moderate values of wave numbers (Figure 2 (a)−(b));
- erroneous results for wave number to 50 (Figure 2 (c));
- dissipated wave when wave number is 100 (Figure 2 (d));
- On the other hand wave obtained using modified scheme gives nodally exact solutions for all values of wave numbers.

4 Modified Compact Explicit and Implicit Schemes for Two dimensional Case

Consider the two dimensional wave equation given by

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad (x,y)\in(a,b) \text{ and } t>0. \tag{27}$$

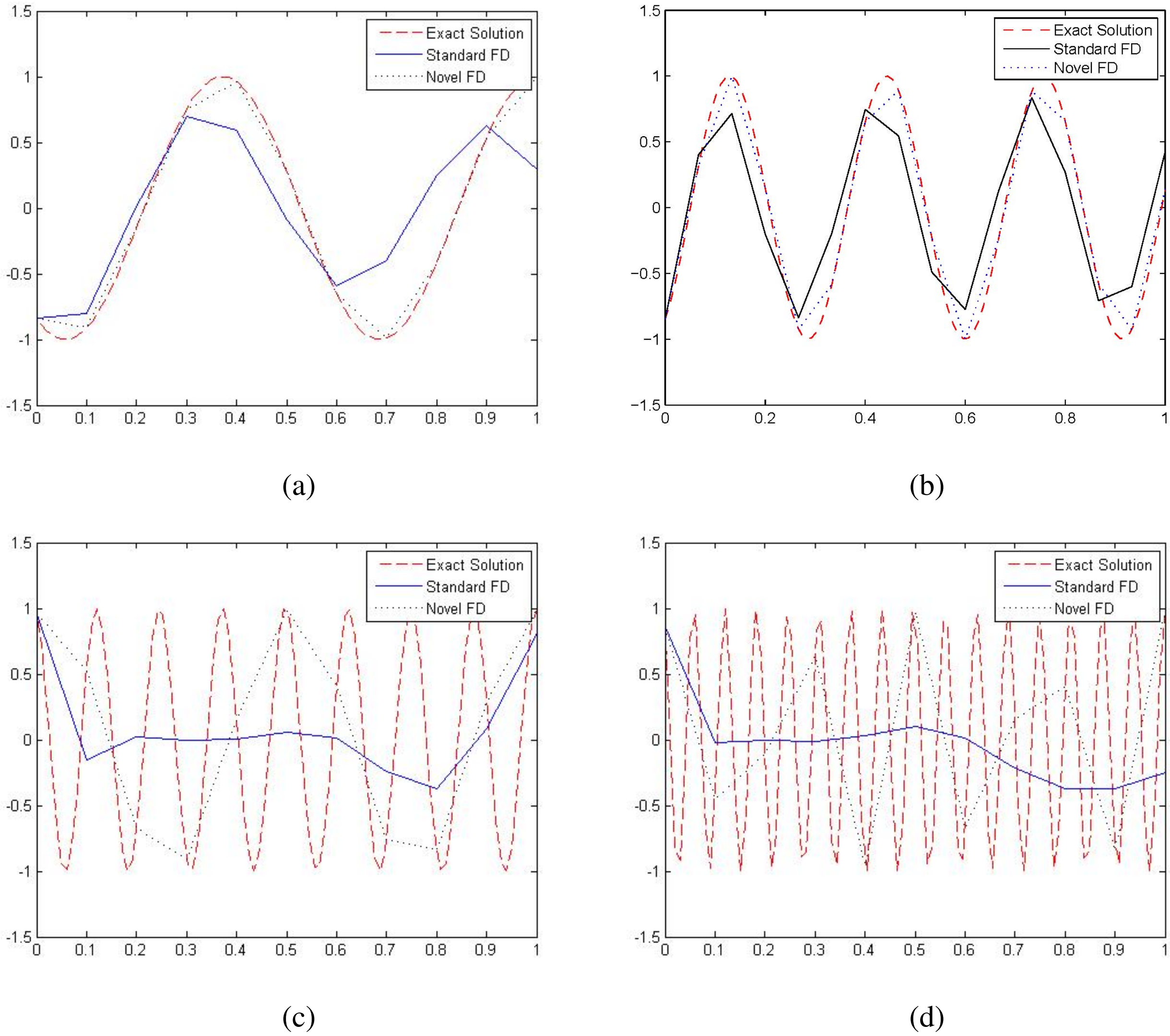


Figure 2: Wave propagation with $r = 0.5$ for (a) $\kappa = 10$, (b) $\kappa = 20$, (c) $\kappa = 50$, and (d) $\kappa = 100$.

In order to construct modified compact explicit scheme, the partial derivatives, $\partial^2 u/\partial x^2, \partial^2 u/\partial y^2$ and $\partial^2 u/\partial t^2$, present in (27) are replaced with below given approximation with α chosen as coefficient of the middle node instead of 2 given by

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{1}{h^2}\left[u^n_{i-1,j} - \alpha u^n_{i,j} + u^n_{i+1,j}\right] + O(h^2),\\
\frac{\partial^2 u}{\partial y^2} &= \frac{1}{h^2}\left[u^n_{i,j-1} - \alpha u^n_{i,j} + u^n_{i,j+1}\right] + O(h^2),\\
\text{and} \quad \frac{\partial^2 u}{\partial t^2} &= \frac{1}{\ell^2}\left[u^{n-1}_{i,j} - \alpha u^n_{i,j} + u^{n+1}_{i,j}\right] + O(\ell^2).
\end{aligned}$$

Substituting above approximations in (27) and performing simplifications gives

$$u^{n+1}_{i,j} = (cr)^2\left(u^n_{i-1,j} + u^n_{i+1,j} - \alpha u^n_{i,j} + u^n_{i,j-1} + u^n_{i,j+1}\right) - u^{n-1}_{i,j}. \qquad (28)$$

Now the value of α is calculated by inserting $u_{i,j}^n = e^{i(h(ik_1+jk_2)-\omega n\ell)}$ into (28)

$$\alpha = \frac{2}{(2(cr)^2-1)}\left[(cr)^2\left(\cos(k_1h)+\cos(k_2h)\right)-\cos(\omega\ell)\right]$$

with $k_1 = k\cos(\theta)$ and $k_2 = k\sin(\theta)$ satisfying $k^2 = k_1^2 + k_2^2$ and θ is the incident angle. Inserting this value of α in (28), gives required modified compact explicit scheme for two dimensional case

$$\begin{aligned} u_{i,j}^{n+1} = (cr)^2\left(u_{i-1,j}^n + u_{i+1,j}^n + u_{i,j-1}^n + u_{i,j+1}^n\right) + \\ \frac{2u_{i,j}^n}{(1-2(cr)^2)}\left[(cr)^2\left(\cos(k_1h)+\cos(k_2h)\right)-\cos(\omega\ell)\right] - u_{i,j}^{n-1}. \end{aligned} \tag{29}$$

Similarly, modified compact implicit scheme for two dimensional case is given by

$$\begin{aligned} \frac{(cr)^2}{4}\left[u_{i-1,j}^{n+1}+u_{i+1,j}^{n+1}+u_{i,j-1}^{n+1}+u_{i,j+1}^{n+1}\right] = -\frac{(cr)^2}{2}\left[u_{i-1,j}^{n}+u_{i+1,j}^{n}+u_{i,j-1}^{n}+u_{i,j+1}^{n}\right] \\ \frac{(cr)^2}{4}\left[u_{i-1,j}^{n-1}+u_{i+1,j}^{n-1}+u_{i,j-1}^{n-1}+u_{i,j+1}^{n-1}\right] \\ +\left[1+\frac{(cr)^2}{2}\left(\cos(k_1h)+\cos(k_2h)\right)+\frac{1-\cos(\omega\ell)}{1-\cos(\omega\ell)}\right]\left(u_{i,j}^{n-1}+u_{i,j}^{n+1}\right) \\ -\left[2-(cr)^2\left(\cos(k_1h)+\cos(k_2h)\right)-\frac{2(1-\cos(\omega\ell))}{(cr)^2(1-\cos(\omega\ell))}\right]u_{i,j}^n. \end{aligned} \tag{30}$$

4.1 Dispersion Analysis

Now to avoid repetition, following steps as presented in Section 4.4 inserting a plane wave solution of the form $u_{i,j}^n = e^{i(\tilde{k_1}ih+\tilde{k_2}jh-\omega n\ell)}$, with $\tilde{k_1}$, $\tilde{k_2}$ as discrete wave numbers into either (29) and (30) results into following after straight forward manipulations

$$[\cos(\tilde{k_1}h)-\cos(k_1h)] + [\cos(\tilde{k_2}h)-\cos(k_2h)] = 0$$

which implies

$$\tilde{k_1} = k_1 \text{ and } \tilde{k_2} = k_2.$$

5 Conclusions

In this work range of modified compact central finite difference (C.F.D.) schemes are constructed in case of rectangular grid for one dimensional transient wave equation. Salient features of these schemes are given below:

1. they provide compact stencil i,e. minimum number of grid nodes are involved;

2. they provide dispersion free numerical results;

3. they do not require to write brand new code altogether which means implementation of these schemes bears no additional cost;

4. cover all range of wave numbers, small as well as large. However, these schemes are constructed for large wave number applications;

5. are computationally attractive and cost effective as achieving optimal results do not require use of fine mesh;

6. leads to standard C.F.D. schemes with series expansion of terms $\cos(\kappa h)$ and $\cos(\omega\ell)$ present in modified schemes;

7. provide highly accurate results even for values of $r > 1$ which is very attractive feature for practical applications.

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