Central generalized bi-semi-derivations on semiprime rings

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Abstract

In this research, our goal is to characterize the structure of central generalized bi-semiderivation δ on ring. Infact, we obtain a few commutativity observations for bi-semi-derivations that commute on prime and semiprime ring. A non-commutative version of some results is also investigated with the help of algebraic identities in which δ will acting as left centralizer.

keywords

Semiprime (prime) ring; algebraic identities; generalized bi-semi-derivation.

1 Introduction

A mapping D from $R\times R$ to R is considered to be symmetric if $D(a,b)=D(b,a)$ for each $a,b\in R$. If D is additive within both slots, it is referred to as bi-additive. We are discussing the conceptual framework of symmetric bi-derivations, as seen in [1], it follows: A mapping $D: R\times R\longrightarrow R$ is referred as bi-derivation if D is bi-additive and for every $a\in R$, the map $b\mapsto D(a,b)$ as well as for every $b\in R$, the map $a\mapsto D(a,b)$ is a derivation of R. For ideational reading in the related matter one can turn to [1]. For a symmetric biadditive mapping D, a map h on R identified as $h(j)=D(j,j)$, for every j in R is commonly referred to the trace of D.

Bergen [2] outlined the idea of semi-derivations on ring R. If a function $g: R\longrightarrow R$ is in existence such that $f(ce)=f(c)g(e)+cf(e)=f(c)e+g(c)f(e)$ and $f(g(e))=g(f(e))$ for any additive mapping f on R, then such map f is termed a semi-derivation. for each $e,c\in R$. Every semiderivations connected to g are manifestly typical derivations, if g is an identity map of R.

A function $\vartheta: R\times R\longrightarrow R$ with symmetry and bi-additivity is termed as symmetric bi-semi-derivation linked with the function f from R to R, if ϑ and f fulfilling the requirements listed below

$$\vartheta(ei,c)=\vartheta(e,c)f(i)+e\vartheta(i,c)=\vartheta(e,c)i+f(e)\vartheta(i,c)$$

$$\vartheta(c,de)=\vartheta(c,d)f(e)+d\vartheta(c,e)=\vartheta(c,d)e+f(d)\vartheta(c,e)$$

and $\vartheta(f)=f(\vartheta)$ for all $d,e,i,c\in R$.

Example 1.1 *Assume that S is a ring after matrix addition and multiplication are applied, where* $S=\left\{\begin{pmatrix} l & k\\ 0 & 0\end{pmatrix} \mid l,k\in R\right\}$ *and a commutative ring R. Define* $\vartheta: S\times S\longrightarrow S$ *such that* $\vartheta\left(\begin{pmatrix} l & k\\ 0 & 0\end{pmatrix},\begin{pmatrix} i & j\\ 0 & 0\end{pmatrix}\right)=\begin{pmatrix} 0 & kj\\ 0 & 0\end{pmatrix}$ *and* $f: S\longrightarrow S$ *by* $f\left(\begin{pmatrix} l & k\\ 0 & 0\end{pmatrix}\right)=\begin{pmatrix} l & 0\\ 0 & 0\end{pmatrix}$. *From this,* ϑ *refers to the bi-semi-derivation on S with associated function* f.

Assume that f is a semi-derivation of R having an associated function g, an endomorphism. A generalized semiderivation is the additive map F on R if $\mathrm{F}(ce)=\mathrm{F}(c)e+g(c)f(e)=\mathrm{F}(c)g(e)+cf(e)$ and $\mathrm{F}(g)=g(\mathrm{F})$, for every c,e in R. There is a generalized semiderivation for every given semi-derivation. Furthermore, any generalized semi-derivations connected to g are just generalized derivations of R, if g is acting as identity on R. The most natural example of generalized semi-derivation, we consider a semi-derivation $\mathscr{F}$ on a ring R joint with a function $\mathscr{G}$ and define the two map as $\mathrm{F}(l)=\mathscr{F}(l)-l$ and $\mathrm{H}(l)=\mathscr{F}(l)+l$, l in R. With such construction the generalized semiderivations on R are represented by F and H joint with $\mathscr{G}$.

Motivated by all above definitions and references includes in [3, 4, 2, 5, 6, 7], we deliver the concept of generalized bi-semiderivation on ring in [8] as follows: Consider the maps $\delta, \vartheta : R \times R \longrightarrow R$ and f from R to R. Now describe if for every $l \in R$, $b \mapsto \delta(l,b)$ and for every $b \in R$, $l \mapsto \delta(l,b)$ are generalized semi-derivation of R with associated function ϑ, f (defined as above), and satisfying $\delta(f) = f(\delta)$, then δ will be called generalized bi-semi-derivation on R. More precisely, δ, ϑ, f satisfying the following:

1. $\delta(lv,c) = \delta(l,c)f(v) + l\vartheta(v,c) = \delta(l,c)v + f(l)\vartheta(v,c)$
2. $\delta(l,wc) = \delta(l,w)f(c) + w\vartheta(l,c) = \delta(l,w)c + f(w)\vartheta(l,c)$
3. $f(\delta) = \delta(f)$ for every $l,v,c,w \in R$.

We present the example of generalized bi-semiderivation to understand the concept well.

Example 1.2 *Consider the set* $\mathscr{R} = \left\{ \begin{pmatrix} l & j \\ u & q \end{pmatrix} \mid j,l,u,q \in 2\mathrm{Z}_8 \right\}$. *Then* $\mathscr{R}$ *represents a ring under matrix addition and matrix multiplication. Define* $\delta, \vartheta : \mathscr{R} \times \mathscr{R} \longrightarrow \mathscr{R}$ *such as*

$$\delta\left(\begin{pmatrix} l & j \\ u & q \end{pmatrix}, \begin{pmatrix} e & k \\ g & h \end{pmatrix} \right) = \begin{pmatrix} 0 & jk \\ ug & 0 \end{pmatrix},$$

$$\vartheta\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & k \\ g & h \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & dh \end{pmatrix}$$

and $f : \mathscr{R} \longrightarrow \mathscr{R}$ *by* $f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$. *Therefore,* δ *is a generalized bi-semiderivation with associated function* ϑ *and* f *on* $\mathscr{R}$. *In above defined concept of generalized bi-semiderivation, we easily observe that* δ *will be considered as bi-semiderivation, if we assume* $\delta = \vartheta$.

Several mathematicians have identified a connection between the behavior of mappings observing algebraic identities involving prime (semiprime) rings and their subsets. The prime ring's R structure possessing a nonzero derivation $\mathscr{D}$ that allows the values of $\mathscr{D}$ to commute, or for which $\mathscr{D}(k)\mathscr{D}(j) = \mathscr{D}(j)\mathscr{D}(k)$ for each $k, j \in R$, was found by Herstein [6]. In this note, author determine the structure of commuting derivation on ring.

In [4], authors goal is to provide commutativity results for rings and show that if I is a nonzero ideal of R and R is a 2-torsion free semiprime ring, then a derivation d of R is commuting on I if one of the following rules is true: (i) $d(x)d(y) = xy$ (ii)$d(x)d(y) = yx$ (iii)$d(x)d(y) = -xy$ (iv)

$d(x)d(x) = x^2$ for all $x, y \in I$. Further, if $d(I) \neq 0$, then R has a nonzero, central ideal. Encouraged by each work of examined literature, we investigate the central generalized bi-semiderivations and it's related identities on semiprime ring.

Lastly, we show by some examples that the limitations placed on the hypothesis of the different theorems are not redundant.

2 Main Results

With the following lemmas, we get started.

Lemma 2.1 *[9] Let R be a semiprime ring, then:*
(1) There are no non-zero nilpotent elements in the center of R.
(2) If for all $u, v \in R$ and a nonzero prime ideal J of R such that $uRv \subseteq J$, then either $u \in J$ or $v \in J$.

Lemma 2.2 *[10] The center of a one sided (non-zero) ideal exists in the center of R, if R is a semiprime ring. Each commutative ideal (one-sided) is therefore always included in the center of R.*

Theorem 2.1 *Let R be a semiprime ring possessing 2 torsion freeness and δ be a generalized bi-semiderivation on R with associated surjective function f and bi-semiderivation ρ. If $\delta(r,r) \subseteq Z(R)$ for every r in R, then either $\rho = 0$ or R contains a central nonzero ideal.*

Proof: We have given that $\delta(r,r) \subseteq Z(R)$, for each r in R. So,

$$[\delta(r,r), s] = 0 \text{ for every } r, s \in R. \tag{1}$$

On linearization of above equation in r and utilize the torsion of R, we observe that

$$[\delta(r,p), s] = 0 \text{ for every } r, p, s \in R. \tag{2}$$

Now, put pt in place of p in (2)to find

$$\delta(r,p)[f(t), s] + p[\rho(r,t), s] + [p, s]\rho(r,t) = 0 \text{ for each } r, t, p, s \in R. \tag{3}$$

Surjectivity of f enable us to put q for $f(t)$, $q \in R$ in (3) and we get

$$\delta(r,p)[q, s] + p[\rho(r,t), s] + [p, s]\rho(r,t) = 0 \text{ for every } r, q, p, s, t \in R. \tag{4}$$

Reinstate (4) after putting ps for s and applying (4) to bring out

$$\delta(r,p)[q,p]s + p[\rho(r,t),p]s = 0 \text{ for each } r,p,t,q,s \in R. \tag{5}$$

At instance, we can find from (5)

$$p[\rho(r,t),p]s = 0 \text{ for every } t,p,s,r \in R. \tag{6}$$

Now make use of semiprimeness of R to obtain $[\rho(r,t),p] = 0$, for every $p,r,t \in R$. Therefore, $\rho(R,R) \subseteq Z(R)$. We conclude our claim by Lemma 2.2 in [11].

Theorem 2.2 *Suppose that I is a nonzero ideal of R and that R is a semiprime ring. If δ is a symmetric generalized bi-semiderivation on R with associated bi-semiderivation ρ and associated function f such that $[\delta(l,l),j] \mp [l,j] = 0$ for all $l,j \in R$, then δ is central. Moreover, either $\rho = 0$ or R contains a central (nonzero) ideal.*

Proof: In accordance to the stated hypothesis, we have

$$[\delta(l,l),j] \mp [l,j] = 0 \text{ for each } l,j \in I. \tag{7}$$

As a result of linearization in l of (7)

$$[\delta(l,l),j] + [\delta(q,q),j] + 2[\delta(l,q),j] \mp [l,j] \mp [q,j] = 0 \text{ for any } l,j,q \in I. \tag{8}$$

Comparing (7) and (8), we obtain

$$[\delta(l,q),j] = 0 \text{ for all } l,j,q \in I. \tag{9}$$

Replace rj in (25) to get $[\delta(l,q),r]j = 0$ for each $l,q,j \in I$ and $r \in R$. This implies that Hence $\delta(l,q) \subseteq Z(R)$. Applying Theorem 2.1, to conclude the proof.

Theorem 2.3 *Let a ring R be prime possessing characteristic not 2. If δ is a symmetric generalized bi-semiderivation on R with associated bi-semiderivation ρ and associated function f such that $[\delta(x,x),y] \mp (x \circ y) = 0$ for all $x,y \in R$, then δ is central δ is central. Moreover, either $\rho = 0$ or R contains a central nonzero ideal.*

Proof: The proof of this theorem is similar as that of above theorem.

Theorem 2.4 *Let R be a semiprime ring and $I \neq (0)$ be an ideal of R. If δ is a symmetric generalized bi-semiderivation on R with associated function f such that $\delta(p,p) \circ y - [p,y] = 0$ for $p,y \in R$, then δ is central. Moreover, either $\rho = 0$ or R contains a central nonzero ideal.*

Proof: In accordance to the stated hypothesis, we have

$$\delta(p,p)\circ y-[p,y]=0 \text{ for each } p,y\in I. \tag{10}$$

Substitution of yz in place of y in (10) gives that

$$[\delta(p,p),y]z+y(\delta(p,p)\circ z)-[p,y]z-y[p,z]=0 \text{ for each } y,p,z\in I. \tag{11}$$

Analyzing the last pair of equations, we have

$$[\delta(p,p),y]z-[p,y]z=0 \text{ for each } y,p,z\in I. \tag{12}$$

Rewrite last expression by putting yx for y to find

$$y[\delta(x,x),x]z=0 \text{ for each } x,y,z\in I. \tag{13}$$

$R's$ semiprimeness indicates that $[\delta(x,x),x]=0$, for every $x\in R$. Hence $\delta(x,z)\subseteq Z(I)\subseteq Z(R)$, utilizing the property that the center of R contains the center of a nonzero ideal by Lemma 2.1.

Theorem 2.5 *Let a semiprime ring be R possessing 2-torsion freeness and $I\neq 0$ be an ideal of R. If δ is a symmetric generalized bi-semiderivation on R with associated function f and associated bi-semiderivation ρ such that $\delta(x,x)\delta(y,y)=xy$ $\forall y,x\in R$, then ρ is central. Moreover, either $\rho=0$ or R contains a central nonzero ideal.*

Proof: We have given that

$$\delta(x,x)\delta(y,y)=xy \text{ for each } x,y\in I. \tag{14}$$

Linearizing (14) in x yields that

$$\delta(x,z)\delta(y,y)=0 \text{ for every } x,z,y\in I. \tag{15}$$

A similar way of linearization of (14) in y give us Linearizing (14) in x yields that

$$\delta(x,x)\delta(y,u)=0 \text{ for each } x,u,y\in I. \tag{16}$$

Substitute xr for x in (15) to get

$$\delta(x,z)r\delta(y,y)+f(x)\rho(r,z)\delta(y,y)=0 \text{ for every } x,z,y\in I,r\in R. \tag{17}$$

Multiply (17) by $\delta(w,w)$ from left and use (16) to find

$$\delta(w,w)f(x)\rho(r,z)\delta(y,y)=0 \text{ for every } w,z,y\in I, r\in R. \tag{18}$$

Applying the surjectivity of f, the last equation can be seen as

$$\delta(w,w)t\rho(r,z)\delta(y,y)=0 \text{ for each } w,z,y\in I, r\in R \text{ and } f(x)=t\in R. \tag{19}$$

This expressly implies that

$$\rho(r,z)\delta(w,w)t\rho(r,z)\delta(w,w)=0 \text{ for every } w,z\in I, r,t\in R. \tag{20}$$

We obtain by $R's$ semiprimeness

$$\rho(r,z)\delta(w,w)=0 \text{ for every } w,z\in I, r\in R. \tag{21}$$

Multiply above equation by $\delta(u,u)$ from right and use (14) to find

$$\rho(r,z)wu=0 \text{ for each } w,u,z\in I, r\in R. \tag{22}$$

A suitable replacement in the last equation enable us to write $[\rho(r,z),u]=0\ \forall r\in R$ and $u,z\in I$. On implementing Lemma 2.2, $\rho(z,z)\subseteq Z(R)$, and hence ρ is central.

Another claim that either $\rho=0$ or R contains a central nonzero ideal can conclude by Lemma 2.2 in [11], if ρ is central.

Theorem 2.6 *Let a ring R be semiprime having 2-torsion freeness and $I\neq 0$ be an ideal of R. If δ is a symmetric generalized bi-semiderivation on R linked with function f and associated bi-semiderivation ρ such that $\delta(x,x)\delta(y,y)=-yx\ \forall y,x\in R$, then ρ is central. Moreover, either $\rho=0$ or R contains a central nonzero ideal.*

Proof: The proof of this theorem is obtained by following the identical approach as in the previous theorem.

The non-commutative version of our previous investigation can be seen as below result, in which we observe that δ will be acting as left centralizer. The detailed concept of left (right) centralizers can be found in [12].

Corollary 2.1 *Letting R be a prime ring that is non-commutative with $char(R)\neq 2$. If δ is a symmetric generalized bi-semiderivation on R with associated function f and associated bi-*

semiderivation ρ such that $\delta(x,x)\delta(y,y) = xy$ for all $x,y \in R$, then $\rho = 0$. In this case, δ will acting as left centralizer.

Theorem 2.7 *Let a ring R be prime having characteristic not 2. If δ is a symmetric generalized bi-semiderivation on R with associated function f such that $[\delta(k,k),u] \mp [\rho(u,u),k] = 0\ \forall u,k \in R$, then δ is central. Moreover, either $\rho = 0$ or R contains a central nonzero ideal.*

Proof: From stated hypothesis, we have

$$[\delta(k,k),u] \mp [\rho(u,u),k] = 0 \text{ for every } u,k \in I. \tag{23}$$

Linearize in k (23) yields that

$$[\delta(k,k),u] + [\delta(v,v),u] + 2[\delta(k,v),u] \mp [\rho(u,u),k] \mp [\rho(u,u),v] = 0 \text{ for every } u,k,v \in I. \tag{24}$$

Comparing (23) and (24) and applying characteristic condition, we obtain

$$[\delta(k,v),u] = 0 \text{ for each } v,u,k \in I. \tag{25}$$

Hence $\delta(k,v) \subseteq Z(I) \subseteq Z(R)$, as follows from Lemma 2.1.

Theorem 2.8 *Let R be a 2-torsion free semiprime ring. If ϑ is a symmetric bi-semiderivation on R such that $\vartheta(\vartheta(u,u),u) = 0$ for all $u \in R$, then $\vartheta = 0$.*

Proof: We are given that by hypothesis

$$\vartheta(\vartheta(u,u),u) = 0 \text{ for all } u \in R. \tag{26}$$

Linearize (26) to obtain

$$\begin{aligned}&\vartheta(\vartheta(u,u),u) + \vartheta(\vartheta(v,v),u) + 2\vartheta(\vartheta(u,v),u) + \vartheta(\vartheta(u,u),v)\\ &+\vartheta(\vartheta(v,v),v) + 2\vartheta(\vartheta(u,v),v) = 0 \text{ for every } v,u \in R.\end{aligned} \tag{27}$$

From (26) and (27), we get

$$\vartheta(\vartheta(v,v),u) + 2\vartheta(\vartheta(u,v),u) + \vartheta(\vartheta(u,u),v) + 2\vartheta(\vartheta(u,v),v) = 0 \text{ for each } v,u \in R. \tag{28}$$

Put $-u$ in place of u in(28) to find

$$-\vartheta(\vartheta(v,v),u) + 2\vartheta(\vartheta(u,v),u) + \vartheta(\vartheta(u,u),v) - 2\vartheta(\vartheta(u,v),v) = 0 \text{ for every } v,u \in R. \tag{29}$$

Analyzing (28) and (29) to obtain by using torsion of R

$$2\vartheta(\vartheta(u,v),v)+\vartheta(\vartheta(v,v),u)=0 \text{ for every } v,u\in R. \tag{30}$$

Rewrite (30) after swapping u by tu, we have

$$\begin{aligned}&2t\vartheta(\vartheta(u,v),v)+2\vartheta(\vartheta(t,v),v)u+2\vartheta(t,v)\vartheta(u,v)+2\vartheta(t,v)\vartheta(u,v)\\&+\vartheta(\vartheta(v,v),t)u+t\vartheta(\vartheta(v,v),u)=0 \text{ for every } u,t,v\in R.\end{aligned} \tag{31}$$

Using (30), (31) becomes

$$4\vartheta(t,v)\vartheta(u,v)=0 \text{ for every } u,t,v\in R. \tag{32}$$

Torsion restriction on R yields that $\vartheta(t,v)\vartheta(u,v)=0$ for each $v,t,u\in R$. In particular, last expression can be written as $\vartheta(t,v)\vartheta(t,v)=0$ for each $v,t\in R$. This implies that $(\vartheta(t,v))^2=0\ \forall t,v\in R$, that is, ϑ is nilpotent with index 2. Use Lemma 2.1 to observe $\vartheta(t,v)=0$ for each $t,v\in R$. Hence $\vartheta=0$.

Example 2.1 *Consider the set* $R=\left\{\begin{pmatrix} s & c\\ t & 0\end{pmatrix} \mid s,c,t\in Z_8\right\}$ *and* $I=\left\{\begin{pmatrix} l & 0\\ j & 0\end{pmatrix} \mid l,j\in Z_8\right\}$. *When performing "+" and "." in matrices of R, R denotes a ring. and I will be a left ideal of R. Define* $\delta: R\times R\longrightarrow R$ *such that* $\delta\left(\begin{pmatrix} s & c\\ t & 0\end{pmatrix},\begin{pmatrix} u & f\\ g & 0\end{pmatrix}\right)=\begin{pmatrix} su & 0\\ 0 & 0\end{pmatrix}$, $\vartheta\left(\begin{pmatrix} s & c\\ t & 0\end{pmatrix},\begin{pmatrix} u & f\\ g & 0\end{pmatrix}\right)=\begin{pmatrix} 0 & 0\\ tg & 0\end{pmatrix}$ *and* $f: R\longrightarrow R$ *by* $f\left(\begin{pmatrix} s & c\\ t & 0\end{pmatrix}\right)=\begin{pmatrix} 0 & 0\\ t & 0\end{pmatrix}$. *Therefore,* δ *is a generalized bi-semiderivation with associated function* ϑ *and associated map* f *on I. We easily observe that the maps* δ,ϑ,f *satisfying the condition of Theorem 2.7 and 2.8 but neither* δ *is central nor* $\vartheta=0$. *Hence the Semiprimeness (Primeness) of ring is the essential requirement of the hypothesis.*

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