Geometry of Quasi bi-slant conformal Submersions from Kenmotsu manifold

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Abstract

In this study, we examine Quasi bi-slant conformal submersions originating from a Kenmotsu manifold, focusing on the vertical Reeb vector field ξ. Initially, we explore the integrability conditions for the distributions defined by quasi-bi-slant submersions. Furthermore, we delve into the geometry of the associated leaves. The research concludes by presenting two intriguing observations regarding the pluriharmonicity of Quasi Bi-Slant Conformal Submersions and includes several non-trivial examples of such submersions.

keywords

Kenmotsu manifold, Riemannian submersions, Conformal bi-slant submersions, quasi bi-slant submersions.

1 Introduction

Immersions and submersions play crucial roles in differential geometry, with slant submersions being a particularly intriguing subject in the fields of differential, complex, and contact geometry. The study of Riemannian submersions between Riemannian manifolds was first explored by O'Neill [24] and Gray [12], independently, and subsequently led to investigations of Riemannian submersions between almost Hermitian manifolds, known as almost Hermitian submersions, by Watson in 1976 [38]. Riemannian submersions have many applications in mathematics and physics, especially in Yang-Mills theory ([7], [39]) and Kaluza-Klein theory ([18], [22]).

Semi-invariant submersions, a generalization of holomorphic submersions and anti-invariant submersions, were introduced by Sahin in 2013 [32]. In 2016, Tatsan, Sahin, and Yanan studied hemi-slant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds, and presented several decomposition theorems for them [37]. R. Prasad et al. further examined quasi bi-slant submersions from almost contact metric manifolds onto Riemannian manifolds [26], as well as from Kenmotsu manifolds [27], which represents a step forward in the study of Riemannian submersions.

Since then, many authors have explored different types of Riemannian submersions, including anti-invariant submersions ([4], [31]), slant submersions [10], [33], semi-slant submersions ([16], [25]), and hemi-slant submersions ([36], [1]), from both almost Hermitian manifolds and almost contact metric manifolds. These studies have greatly expanded our understanding of the geometrical structures of Riemannian manifolds.

The concept of almost contact Riemannian submersions from almost contact manifold was introduced by Chinea in [8]. Chinea also examined the fibre space, base space and total space using a differential geometric perspective. To generalize Riemannian submersions, Gundmundsson and Wood [14, 15] presented horizontally conformal submersion, defined as: Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds of dimension m_1 and m_2, respectively. A smooth map $J : (M_1, g_1) \to (M_2, g_2)$ is called a horizontally conformal submersion, if there is a positive function λ such that

$$\lambda^2 g_1(X_1, X_2) = g_2(J_* X_1, J_* X_2), \tag{1.1}$$

for all $X_1, X_2 \in \Gamma(kerJ)^\perp$. Thus, Riemannian submersion is a particular horizontally conformal submersion with $\lambda = 1$. Later on, Fuglede [13] and Ishihara [20] separately studied horizontally conformal submersions. Additionally various other kind of submersions such as, conformal slant submersions [3], conformal anti-invariant submersions [34], conformal semi-slant submersions [2], conformal semi-invariant submersions [5] and conformal anti-invariant submersions [27] have been studied by Akyol and Sahin and R. Prasad et al [28]. Furthermore, Shuaib and Fatima recently explored conformal hemi-slant Riemannian submersions from almost product manifolds onto Riemannian manifolds [35].

In this paper, we study quasi bi-slant conformal submersions from Kenmotsu manifold onto a Riemannian manifold considering the Reeb vector field ξ vertical. This paper is divided into six sections. Section 2 contains definitions of almost contact metric manifolds and, in particular, Kenmotsu manifolds. In section 3, fundamental results for quasi bi-slant conformal submersion are investigated, which are necessary our main results. The conditions of integrability and totally geodesicness of distributions are explored in Section 4. Section 5 provides some condition under which a Riemannian submersion becomes totally geodesic as well as some decomposition theorems for quasi bi-slant conformal submersion are obtained. The last section discusses ϕ-pluriharmonicity.

Note: Throughout the paper, we will consider abbreviations as follows:
Riemannian submersion- RS, Riemannian Manifold- RM, Almost contact metric manifold-ACM manifold, Quasi bi-slant conformal submersion- $\mathscr{QBSCS}$, gradient- G.

2 Preliminaries

Let M be a $(2n+1)$-dimensional almost contact manifold with almost contact structures (ϕ,ξ,η), where a $(1,1)$ tensor field ϕ, a vector field ξ and a 1-form η satisfying

$$\phi^2 = -I+\eta\otimes\xi,\ \ \phi\xi = 0,\ \ \eta\circ\phi = 0,\ \ \eta(\xi) = 1, \tag{2.2}$$

where I is the identity tensor. The almost contact structure is said to be normal if $N+d\eta\otimes\xi = 0$, where N is the Nijenhuis tensor of ϕ. Suppose that a Riemannian metric tensor g is given in M and satisfies the condition

$$g(\phi\widehat{U},\phi\widehat{V}) = g(\widehat{U},\widehat{V})-\eta(\widehat{U})\eta(\widehat{V}),\ \ \eta(\widehat{U}) = g(\widehat{U},\xi). \tag{2.3}$$

Then (ϕ,ξ,η,g)-structure is called an almost contact metric structure. Define a tensor field Φ of type $(0,2)$ by $\Phi(\widehat{X},\widehat{Y}) = g(\phi\widehat{X},\widehat{Y})$. If $d\eta = \Phi$, then an almost contact metric structure is said to be normal contact metric structure. Let Φ be the fundamental 2-form on M, i.e, $\Phi(\widehat{U},\widehat{V}) = g(\widehat{U},\phi\widehat{V})$. If $\Phi = d\eta$, M is said to be a contact manifold. If ξ is a Killing vector field with respect to g, the contact metric structure is called a K-contact structure.

S.Tanno [30], who categorized connected, almost contact metric manifolds with the largest automorphism groups. The sectional curvature of a plane section containing ξ for such a manifold is a constant c. This classification includes classes of warped products with $c<0$ is $R\times_f C^n$. The tensorial equation of these manifolds are:

$$(\nabla_{\widehat{U}}\phi)\widehat{V} = g(\phi\widehat{U},\widehat{V})\xi-\eta(\widehat{V})\phi\widehat{U}. \tag{2.4}$$

Kenmotsu [21], investigated a few basic differential geometric features of these spaces, giving rise to the name Kenmotsu manifolds. It is also apparent on a Kenmotsu manifold M that

$$\nabla_{\widehat{U}}\xi = -\phi^2\widehat{U} = \widehat{U} - \eta(\widehat{U})\xi, \tag{2.5}$$

The covariant derivative of ϕ is defined by

$$(\nabla_{\widehat{U}_1}\phi)\widehat{V}_1 = \nabla_{\widehat{U}_1}\phi\widehat{V}_1 - \phi\nabla_{\widehat{U}_1}\widehat{V}_1, \tag{2.6}$$

for any vector fields $\widehat{U}_1, \widehat{V}_1 \in \Gamma(TM)$. Now we outline conformal submersion and examine several relevant results that assist us attain our major goals.

Definition 2.1 *[6] Let J be a Riemannian submersion (RS) from an ACM manifold $(\bar{B}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (RM) $(\bar{B}_2, g_2)$. Then J is called a horizontally conformal submersion, if there is a positive function λ such that*

$$g_1(\widehat{U}_1, \widehat{V}_1) = \frac{1}{\lambda^2} g_2(J_*\widehat{U}_1, J_*\widehat{V}_1), \tag{2.7}$$

for any $\widehat{U}_1, \widehat{V}_1 \in \Gamma(kerJ_*)^\perp$. It is obvious that every RS is a particularly horizontally conformal submersion with $\lambda = 1$.

Let $J : (\bar{B}_1, \phi, \xi, \eta, g_1) \to (\bar{B}_2, g_2)$ be a RS. A vector field $\widehat{X}$ on $\bar{B}_1$ is called a basic vector field if $\widehat{X} \in \Gamma(\ker J_*)^\perp$ and J-related with a vector field $\widehat{X}$ on $\bar{B}_2$ i.e $J_*(\widehat{X}(q)) = \widehat{X}J(q)$ for $q \in \bar{B}_1$.

The formulas provide the two $(1,2)$ tensor fields $\mathscr{T}$ and $\mathscr{A}$ by O'Neill are

$$\mathscr{A}_{E_1}F_1 = \mathscr{H}\nabla_{\mathscr{H}E_1}\mathscr{V}F_1 + \mathscr{V}\nabla_{\mathscr{H}E_1}\mathscr{H}F_1, \tag{2.8}$$

$$\mathscr{T}_{E_1}F_1 = \mathscr{H}\nabla_{\mathscr{V}E_1}\mathscr{V}F_1 + \mathscr{V}\nabla_{\mathscr{V}E_1}\mathscr{H}F_1, \tag{2.9}$$

for any $E_1, F_1 \in \Gamma(T\bar{B}_1)$ and ∇ is Levi-Civita connection of g_1. Note that a RS $J : (\bar{B}_1, \phi, \xi, \eta, g_1) \to (\bar{B}_2, g_2)$ has totally geodesic fibers if and only if $\mathscr{T}$ vanishes identically. From equations (2.8) and (2.9), we can deduce

$$\nabla_{\widehat{U}_1}\widehat{V}_1 = \mathscr{T}_{\widehat{U}_1}\widehat{V}_1 + \mathscr{V}\nabla_{\widehat{U}_1}\widehat{V}_1 \tag{2.10}$$

$$\nabla_{\widehat{U}_1}\widehat{X}_1 = \mathscr{T}_{\widehat{U}_1}\widehat{X}_1 + \mathscr{H}\nabla_{\widehat{U}_1}\widehat{X}_1 \tag{2.11}$$

$$\nabla_{\widehat{X}_1}\widehat{U}_1 = \mathscr{A}_{\widehat{X}_1}\widehat{U}_1 + \mathscr{V}_1\nabla_{\widehat{X}_1}\widehat{U}_1 \tag{2.12}$$

$$\nabla_{\widehat{X}_1}\widehat{Y}_1 = \mathscr{H}\nabla_{\widehat{X}_1}\widehat{Y}_1 + \mathscr{A}_{\widehat{X}_1}\widehat{Y}_1 \tag{2.13}$$

for any vector fields $\widehat{U}_1, \widehat{V}_1 \in \Gamma(kerJ_*)$ and $\widehat{X}_1, \widehat{Y}_1 \in \Gamma(kerJ_*)^\perp$ [11].

It is obvious that $\mathscr{T}$ and $\mathscr{A}$ are skew-symmetric, that is

$$g(\mathscr{A}_{\widehat{X}}E_1,F_1) = -g(E_1,\mathscr{A}_{\widehat{X}}F_1), g(\mathscr{T}_{\widehat{V}}E_1,F_1) = -g(E_1,\mathscr{T}_{\widehat{V}}F_1), \tag{2.14}$$

for any vector fields $E_1,F_1 \in \Gamma(T_p\bar{B}_1)$.

Definition 2.2 *A horizontally conformally submersion $J : \bar{B}_1 \to \bar{B}_2$ is called horizontally homothetic if the gradient (G) of its dilation λ is vertical, i.e.,*

$$H(G\lambda) = 0, \tag{2.15}$$

at $p \in TM_1$, where H is the complement orthogonal distribution to $\nu = kerJ_*$ in $\Gamma(T_pM)$.

The second fundamental form of smooth map J is provided by the formula

$$(\nabla J_*)(\widehat{U}_1,\widehat{V}_1) = \nabla^J_{\widehat{U}_1}J_*\widehat{V}_1 - J_*\nabla_{\widehat{U}_1}\widehat{V}_1, \tag{2.16}$$

and the map be totally geodesic if $(\nabla J_*)(\widehat{U}_1,\widehat{V}_1) = 0$ for all $\widehat{U}_1,\widehat{V}_1 \in \Gamma(T_pM)$ where ∇ and ∇J_* are Levi-Civita and pullback connections.

Lemma 2.1 *Let $J : \bar{B}_1 \to \bar{B}_2$ be a horizontal conformal submersion. Then, we have*

(i) $(\nabla J_*)(\widehat{X}_1,\widehat{Y}_1) = \widehat{X}_1(ln\lambda)J_*(\widehat{Y}_1) + \widehat{Y}_1(ln\lambda)J_*(\widehat{X}_1) - g_1(\widehat{X}_1,\widehat{Y}_1)J_*(grad\ ln\lambda)$,

(ii) $(\nabla J_*)(\widehat{U}_1,\widehat{V}_1) = -J_*(\mathscr{T}_{\widehat{U}_1}\widehat{V}_1)$,

(iii) $(\nabla J_*)(\widehat{X}_1,\widehat{U}_1) = -J_*(\nabla_{\widehat{X}_1}\widehat{U}_1) = -J_*(\mathscr{A}_{\widehat{X}_1}\widehat{U}_1)$

for any horizontal vector fields $\widehat{X}_1,\widehat{Y}_1$ and vertical vector fields $\widehat{U}_1,\widehat{V}_1$ [6].

3 Quasi bi-slant conformal submersions

Definition 3.1 *Let $(\bar{B}_1,\phi,\xi,\eta,g_1)$ be a ACM manifold and $(\bar{B}_2,g_2)$ a Riemannian manifold. A RS $J : \bar{B}_1 \to \bar{B}_2$ is called quasi bi-slant conformal submersion ($\mathscr{QBSCS}$) if there exists mutually orthogonal distributions $\mathfrak{D}$, $\mathfrak{D}_{\theta_1}$ and $\mathfrak{D}_{\theta_2}$ with $kerJ_* = \mathfrak{D} \oplus \mathfrak{D}_{\theta_1} \oplus \mathfrak{D}_{\theta_2}$ where, $\mathfrak{D}$ is invariant under ϕ. i.e., $\phi\mathfrak{D} = \mathfrak{D}$, for the slant distributions $\phi\mathfrak{D}_{\theta_1} \perp \mathfrak{D}_{\theta_1}$, $\phi\mathfrak{D}_{\theta_2} \perp \mathfrak{D}_{\theta_2}$ and for any non-zero vector field $\widehat{V}_i \in (\mathfrak{D}_{\theta_i})_{p_i}, p_i \in \bar{B}_i$ the angle θ_i between $(\mathfrak{D}_{\theta_i})_{p_i}$ and $\phi\widehat{V}_i$ is constant and independent of the choice of the point p_i and $\widehat{V}_i \in (\mathfrak{D}_{\theta_i})_p$, for $i = 1,2$, where θ_1 and θ_2 are called the slant angles of submersion.*

If we suppose m_1, m_2 and m_3 are the dimensions of $\mathfrak{D}$, $\mathfrak{D}_{\theta_1}$ and $\mathfrak{D}_{\theta_2}$ respectively, then we have the following:

(i) If $m_1 \neq 0$, $m_2 = 0$ and $m_3 = 0$, then J is an invariant submersion.

(ii) If $m_1 \neq 0$, $m_2 \neq 0, 0 < \theta_1 < \frac{\pi}{2}$ and $m_3 = 0$, then J is a proper semi-slant submersion.

(iii) If $m_1 = 0$, $m_2 = 0$ and $m_3 \neq 0$, $0 < \theta_2 < \frac{\pi}{2}$, then J is a slant submersion with slant angle θ_2.

(iv) If $m_1 = 0, m_2 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $m_3 \neq 0, \theta_2 = \frac{\pi}{2}$,then J proper hemi-slant submersion.

(v) If $m_1 = 0, m_2 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $m_3 \neq 0$, $0 < \theta_2 < \frac{\pi}{2}$, then J is proper bi-slant submersion with slant angles θ_1 and θ_2.

(vi) If $m_1 \neq 0, m_2 \neq 0$, $0 < \theta_1 < \frac{\pi}{2}$ and $m_3 \neq 0$, $0 < \theta_2 < \frac{\pi}{2}$, then J is proper quasi bi-slant submersion with slant angles θ_1 and θ_2. "

Let J be a $\mathscr{QBSCS}$ from an ACM manifold $(\bar{B}_1, \phi, \xi, \eta, g_1)$ onto a RM $(\bar{B}_2, g_2)$. Then, for any $U \in (kerJ_*)$, we have

$$\widehat{U} = P_1\widehat{U} + P_2\widehat{U} + P_3\widehat{U} \tag{3.17}$$

where P_1, P_2 and P_3 are the projections morphism onto $\mathfrak{D}, \mathfrak{D}_{\theta_1}$, and $\mathfrak{D}_{\theta_2}$. Now, for any $\widehat{U} \in (kerJ_*)$, we have

$$\phi\widehat{U} = \omega\widehat{U} + \chi\widehat{U} \tag{3.18}$$

where $\omega\widehat{U} \in \Gamma(kerJ_*)$ and $\chi\widehat{U} \in \Gamma(kerJ_*)^\perp$. From equations (3.17) and (3.18), we have

$$\begin{aligned}\phi\widehat{U} =& \phi(P_1\widehat{U}) + \phi(P_2\widehat{U}) + \phi(P_3\widehat{U}) \\ =& \omega(P_1\widehat{U}) + \chi(P_1\widehat{U}) + \omega(P_2\widehat{U}) + \chi(P_2\widehat{U}) + \omega(P_3\widehat{U}) + \chi(P_3\widehat{U}).\end{aligned}$$

Since $\phi\mathfrak{D} = \mathfrak{D}$ and $\chi(P_1\widehat{U}) = 0$, we have

$$\phi\widehat{U} = \omega(P_1\widehat{U}) + \omega(P_2\widehat{U}) + \chi(P_2\widehat{U}) + \omega(P_3\widehat{U}) + \chi(P_3\widehat{U}).$$

Hence we have the decomposition as :

$$\phi(kerJ_*) = \omega\mathfrak{D} \oplus \omega\mathfrak{D}_{\theta_1} \oplus \omega\mathfrak{D}_{\theta_2} \oplus \chi\mathfrak{D}_{\theta_1} \oplus \chi\mathfrak{D}_{\theta_2}. \tag{3.19}$$

From equations (3.19), we have the following decomposition

$$(kerJ_*)^\perp = \chi\mathfrak{D}_{\theta_1} \oplus \chi\mathfrak{D}_{\theta_2} \oplus \mu, \tag{3.20}$$

where μ is the orthogonal complement to $\chi\mathfrak{D}_{\theta_1} \oplus \chi\mathfrak{D}_{\theta_2}$ in $(kerJ_*)^\perp$ such that $\mu = (\phi\mu) \oplus < \xi >$ and μ is invariant with respect to ϕ. Now, for any $\widehat{X} \in \Gamma(kerJ_*)^\perp$, we have

$$\phi\widehat{X} = t\widehat{X} + n\widehat{X} \tag{3.21}$$

where $t\widehat{X} \in \Gamma(kerJ_*)$ and $n\widehat{X} \in \Gamma(kerJ_*)^\perp$.

Lemma 3.1 *Let $(\bar{B}_1,\phi,\xi,\eta,g_1)$ be an ACM manifold and $(\bar{B}_2,g_2)$ be a RM. If $J:\bar{B}_1 \to \bar{B}_2$ is a $\mathscr{QBSCS}$, then we have*

$$-\widehat{U}+\eta(\widehat{U})\xi = \omega^2\widehat{U}+t\chi\widehat{U},\ \chi\omega\widehat{U}+n\chi\widehat{U}=0,$$

$$-\widehat{X} = \chi t\widehat{X}+n^2\widehat{X},\ \omega t\widehat{X}+tn\widehat{X}=0,$$

for $\widehat{U} \in \Gamma(kerJ_*)$ and $\widehat{X} \in \Gamma(kerJ_*)^\perp$.

Proof. On using equations (2.2), (3.18) and (3.21), we get the desired results. Since $J:\bar{B}_1 \to \bar{B}_2$ is a $\mathscr{QBSCS}$, Then let us provide some helpful findings that will be utilise throughout the paper.

Lemma 3.2 *Let J be a $\mathscr{QBSCS}$ from an ACM manifold $(\bar{B}_1,\phi,\xi,\eta,g_1)$ onto a RM $(\bar{B}_2,g_2)$, then we have*

(i) $\omega^2\widehat{U} = -\cos^2\theta_1\widehat{U}$,

(ii) $g_1(\omega\widehat{U},\omega\widehat{V}) = \cos^2\theta_1\, g_1(\widehat{U},\widehat{V})$,

(iii) $g(\chi\widehat{U},\chi\widehat{V}) = \sin^2\theta_1\, g_1(\widehat{U},\widehat{V})$,

for any vector fields $\widehat{U},\widehat{V} \in \Gamma(\mathfrak{D}_{\theta_1})$.

Lemma 3.3 *Let J be a $\mathscr{QBSCS}$ from an ACM manifold $(\bar{B}_1,\phi,\xi,\eta,g_1)$ onto a RM $(\bar{B}_2,g_2)$, then we have*

(i) $\omega^2\widehat{Z} = -\cos^2\theta_2\widehat{Z}$,

(ii) $g_1(\omega\widehat{Z},\omega\widehat{W}) = \cos^2\theta_2\, g_1(\widehat{Z},\widehat{W})$,

(iii) $g_1(\chi\widehat{Z},\chi\widehat{W}) = \sin^2\theta_2\, g_1(\widehat{Z},\widehat{W})$,

for any vector fields $\widehat{Z},\widehat{W} \in \Gamma(\mathfrak{D}_{\theta_2})$.

Proof. The proof of the preceding Lemmas is identical to the proof of Theorem (2.2) of [9]. As a result, we omit the proofs.

Let us suppose that $(\bar{B}_2,g_2)$ be a Riemannian manifold and $(\bar{B}_1,\phi,\xi,\eta,g_1)$ be a Kenmotsu manifold. We now analyse how the Kenmotsu structure on $\bar{B}_1$ influences the tensor fields $\mathscr{T}$ and $\mathscr{A}$ of $\mathscr{QBSCS}$ $J:(\bar{B}_1,\phi,\xi,\eta,g_1) \to (\bar{B}_2,g_2)$.

Lemma 3.4 *Let J be a $\mathscr{QBSCS}$ from Kenmotsu manifold $(\bar{B}_1,\phi,\xi,\eta,g_1)$ onto a RM $(\bar{B}_2,g_2)$, then we have*

$$\mathscr{A}_{\widehat{X}}n\widehat{Y}+\mathscr{V}\nabla_{\widehat{X}}t\widehat{Y}=t\mathscr{H}\nabla_{\widehat{X}}\widehat{Y}+\omega\mathscr{A}_{\widehat{X}}\widehat{Y}+g_1(\phi\widehat{X},\widehat{Y})\xi \tag{3.22}$$

$$\mathscr{H}\nabla_{\widehat{X}}n\widehat{Y}+\mathscr{A}_{\widehat{X}}t\widehat{Y}=n\mathscr{H}\nabla_{\widehat{X}}\widehat{Y}+\chi\mathscr{A}_{\widehat{X}}\widehat{Y} \tag{3.23}$$

$$\mathscr{V}\nabla_{\widehat{X}}\omega\widehat{V}+\mathscr{A}_{\widehat{X}}\chi\widehat{V}=t\mathscr{A}_{\widehat{X}}\widehat{V}+\omega\mathscr{V}\nabla_{\widehat{X}}\widehat{V}+g_1(t\widehat{X},\widehat{U})\xi-\eta(\widehat{U})t\widehat{X} \tag{3.24}$$

$$\mathscr{A}_{\widehat{X}}\omega\widehat{V}+\mathscr{H}\nabla_{\widehat{X}}\chi\widehat{V}=n\mathscr{A}_{\widehat{X}}\widehat{V}+\chi\mathscr{V}\nabla_{\widehat{X}}\widehat{V}-\eta(\widehat{U})n\widehat{X} \tag{3.25}$$

$$\mathscr{V}\nabla_{\widehat{V}}t\widehat{X}+\mathscr{T}_{\widehat{V}}n\widehat{X}=\omega\mathscr{T}_{\widehat{V}}\widehat{X}+t\mathscr{H}\nabla_{\widehat{V}}\widehat{X}+g_1(\widehat{X},\phi\widehat{V})\xi \tag{3.26}$$

$$\mathscr{T}_{\widehat{V}}t\widehat{X}+\mathscr{H}\nabla_{\widehat{V}}n\widehat{X}=\chi\mathscr{T}_{\widehat{V}}\widehat{X}+n\mathscr{H}\nabla_{\widehat{V}}\widehat{X} \tag{3.27}$$

$$\mathscr{V}\nabla_{\widehat{U}}\omega\widehat{V}+\mathscr{T}_{\widehat{U}}\chi\widehat{V}-\omega\mathscr{V}\nabla_{\widehat{U}}\widehat{V}=t\mathscr{T}_{\widehat{U}}\widehat{V}+g_1(\phi\widehat{U},\widehat{V})\xi-\eta(\widehat{V})\omega\widehat{U} \tag{3.28}$$

$$\mathscr{T}_{\widehat{U}}\omega\widehat{V}+\mathscr{H}\nabla_{\widehat{U}}\chi\widehat{V}=n\mathscr{T}_{\widehat{U}}\widehat{V}+\chi\mathscr{V}\nabla_{\widehat{U}}\widehat{V}-\eta(\widehat{V})\chi\widehat{U}, \tag{3.29}$$

for any vector fields $\widehat{U},\widehat{V}\in\Gamma(\ker J_*)$ and $\widehat{X},\widehat{Y}\in\Gamma(\ker J_*)^{\perp}$.

Proof. From (3.21), (2.13) and (2.6), we obtained the conditions (3.22) and (3.23). Again using equations (3.18), (3.21), (2.10)-(2.13) and (2.6), finish he result. We will now go through some key conclusions that can be utilized to examine the geometry of $\mathscr{QBSCS}$ $J:\bar{B}_1\to\bar{B}_2$. From the direct calculations, we can conclude the following:

$$(\nabla_{\widehat{U}}\omega)\widehat{V}=\mathscr{V}\nabla_{\widehat{U}}\omega\widehat{V}-\omega\mathscr{V}\nabla_{\widehat{U}}\widehat{V} \tag{3.30}$$

$$(\nabla_{\widehat{U}}\chi)\widehat{V}=\mathscr{H}\nabla_{\widehat{U}}\chi\widehat{V}-\chi\mathscr{V}\nabla_{\widehat{U}}\widehat{V} \tag{3.31}$$

$$(\nabla_{\widehat{X}}t)\widehat{Y}=\mathscr{V}\nabla_{\widehat{X}}t\widehat{Y}-t\mathscr{H}\nabla_{\widehat{X}}\widehat{Y} \tag{3.32}$$

$$(\nabla_{\widehat{X}}n)\widehat{Y}=\mathscr{H}\nabla_{\widehat{X}}n\widehat{Y}-n\mathscr{H}\nabla_{\widehat{X}}\widehat{Y}, \tag{3.33}$$

for any vector fields $\widehat{U},\widehat{V}\in\Gamma(\ker J_*)$ and $\widehat{X},\widehat{Y}\in\Gamma(\ker J_*)^{\perp}$.

Lemma 3.5 *Let $(\bar{B}_1,\phi,\xi,\eta,g_1)$ be a Kenmotsu manifold and $(\bar{B}_2,g_2)$ be a RM. If $J:\bar{B}_1\to\bar{B}_2$ is a $\mathscr{QBSCS}$, then we have*

$$(\nabla_{\widehat{U}}\omega)\widehat{V}=t\mathscr{T}_{\widehat{U}}\widehat{V}-\mathscr{T}_{\widehat{U}}\chi\widehat{V}+g_1(\phi\widehat{U},\widehat{V})\xi-\eta(\widehat{V})\omega\widehat{U}$$

$$(\nabla_{\widehat{U}}\chi)\widehat{V}=n\mathscr{T}_{\widehat{U}}\widehat{V}-\mathscr{T}_{\widehat{U}}\omega\widehat{V}-\eta(\widehat{V})\chi\widehat{U}$$

$$(\nabla_{\widehat{X}}t)\widehat{Y}=\omega\mathscr{A}_{\widehat{X}}\widehat{Y}-\mathscr{A}_{\widehat{X}}n\widehat{Y}+g_1(\phi\widehat{X},\widehat{Y})\xi$$

$$(\nabla_{\widehat{X}}n)\widehat{Y}=\chi\mathscr{A}_{\widehat{X}}\widehat{Y}-\mathscr{A}_{\widehat{X}}t\widehat{Y},$$

for all vector fields $\widehat{U},\widehat{V}\in\Gamma(kerJ_*)$ and $\widehat{X},\widehat{Y}\in\Gamma(kerJ_*)^{\perp}$.

Proof. From some basic facts and taking account the fact from equations (2.6), (2.10)-(2.13) and equations (3.30)-(3.33), we can obtained the results. The tensor fields ω and χ, if they are parallel with regard to the connection ∇ of $\bar{B}_1$, then we obtain

$$t\mathscr{T}_{\widehat{U}}\widehat{V} = \mathscr{T}_{\widehat{U}}\chi\widehat{V},\ n\mathscr{T}_{\widehat{U}}\widehat{V} = \mathscr{T}_{\widehat{U}}\omega\widehat{V}$$

for any vector fields $\widehat{U}, \widehat{V} \in \Gamma(T\bar{B}_1)$.

4 Integrability and totally geodesicness of distributions

Since $(\bar{B}_1, \phi, \xi, \eta, g_1)$ stands for a Kenmotsu manifold and $(\bar{B}_2, g_2)$ for a Riemannian manifold such that $J : \bar{B}_1 \to \bar{B}_2$ is a $\mathscr{QBSCS}$. Three mutually orthogonal distributions, including an invariant distribution $\mathfrak{D}$ and a pair of slant distributions $\mathfrak{D}_{\theta_1}$ and $\mathfrak{D}_{\theta_2}$, are assured by the theory of $\mathscr{QBSCS}$. The integrability of slant distribution is assessed to begin the debate on distributions integrability in the following manner.

Theorem 4.1 *Let J be a $\mathscr{QBSCS}$ from Kenmotsu manifold $(\bar{B}_1, \phi, \xi, \eta, g_1)$ onto a RM $(\bar{B}_2, g_2)$. Then slant distribution $\mathfrak{D}_{\theta_1}$ is integrable if and only if*

$$\begin{aligned} &\frac{1}{\lambda^2}\{g_2(-\nabla^J_{\widehat{U}_1} J_*\chi\widehat{V}_1 + \nabla^J_{\widehat{V}_1} J_*\chi\widehat{U}_1, J_*\chi P_3\widehat{Z})\} \\ &= \frac{1}{\lambda^2}\{g_2((\nabla J_*)(\widehat{U}_1, \chi\widehat{V}_1) + (\nabla J_*)(\widehat{V}_1, \chi\widehat{U}_1), J_*\chi P_3\widehat{Z})\} \\ &\quad - g_1(\nabla_{\widehat{V}_1}\chi\omega\widehat{U}_1 - \nabla_{\widehat{U}_1}\chi\omega\widehat{V}_1, \widehat{Z}) - g_1(\mathscr{T}_{\widehat{U}_1}\chi\widehat{V}_1 - \mathscr{T}_{\widehat{V}_1}\chi\widehat{U}_1, \phi P_1\widehat{Z} + \omega P_3\widehat{Z}), \end{aligned} \tag{4.34}$$

for any $\widehat{U}_1, \widehat{V}_1 \in \Gamma(\mathfrak{D}_{\theta_1})$ and $\widehat{Z} \in \Gamma(\mathfrak{D} \oplus \mathfrak{D}_{\theta_2} \oplus <\xi>)$.

Proof. For all $\widehat{U}_1, \widehat{V}_1 \in \Gamma(\mathfrak{D}_{\theta_1})$ and $\widehat{Z} \in \Gamma(\mathfrak{D} \oplus \mathfrak{D}_{\theta_2} \oplus <\xi>)$ with using equations (2.3), (2.6), (2.4) and (3.18), we get

$$\begin{aligned} g_1([\widehat{U}_1, \widehat{V}_1], \widehat{Z}) = &\ g_1(\nabla_{\widehat{U}_1}\omega\widehat{V}_1, \phi\widehat{Z}) + g_1(\nabla_{\widehat{U}_1}\chi\widehat{V}_1, \phi\widehat{Z}) - g_1(\nabla_{\widehat{V}_1}\omega\widehat{U}_1, \phi\widehat{Z}) \\ &- g_1(\nabla_{\widehat{V}_1}\chi\widehat{U}_1, \phi\widehat{Z}). \end{aligned}$$

By using equations (2.6), (2.4) and (3.18), we have

$$\begin{aligned} g_1([\widehat{U}_1, \widehat{V}_1], \widehat{Z}) = &-g_1(\nabla_{\widehat{U}_1}\omega^2\widehat{V}_1, \widehat{Z}) - g_1(\nabla_{\widehat{U}_1}\chi\omega\widehat{V}_1, \widehat{Z}) + g_1(\nabla_{\widehat{V}_1}\omega^2\widehat{U}_1, \widehat{Z}) \\ &+ g_1(\nabla_{\widehat{V}_1}\chi\omega\widehat{U}_1, \widehat{Z}) + g_1(\nabla_{\widehat{U}_1}\chi\widehat{V}_1, \phi P_1\widehat{Z} + \omega P_3\widehat{Z} + \chi P_3\widehat{Z}) \\ &- g_1(\nabla_{\widehat{V}_1}\chi\widehat{U}_1, \phi P_1\widehat{Z} + \omega P_3\widehat{Z} + \chi P_3\widehat{Z}). \end{aligned}$$

Taking account the fact of Lemma 3.2 with equation (2.11), we get

$$g_1([\widehat{U}_1,\widehat{V}_1],\widehat{Z}) = cos^2\theta_1 g_1([\widehat{U}_1,\widehat{V}_1],\widehat{Z}) + g_1(\nabla_{\widehat{V}_1}\chi\omega\widehat{U}_1 - \nabla_{\widehat{U}_1}\chi\omega\widehat{V}_1,\widehat{Z})$$
$$+ g_1(\mathscr{T}_{\widehat{U}_1}\chi\widehat{V}_1 - \mathscr{T}_{\widehat{V}_1}\chi\widehat{U}_1,\phi P_1\widehat{Z} + \omega P_3\widehat{Z})$$
$$+ g_1(\mathscr{H}\nabla_{\widehat{U}_1}\chi\widehat{V}_1 - \mathscr{H}\nabla_{\widehat{V}_1}\chi\widehat{U}_1,\chi P_3\widehat{Z}).$$

On using equation (2.7), formula (2.16) with Lemma 2.1, we finally get

$$sin^2\theta_1 g_1([\widehat{U}_1,\widehat{V}_1],\widehat{Z})$$
$$= \frac{1}{\lambda^2}\{g_2((\nabla J_*)(\widehat{U}_1,\chi\widehat{V}_1),J_*\chi P_3\widehat{Z}) + g_2((\nabla J_*)(\widehat{V}_1,\chi\widehat{U}_1),J_*\chi P_3\widehat{Z})\}$$
$$+ g_1(\mathscr{T}_{\widehat{U}_1}\chi\widehat{V}_1 - \mathscr{T}_{\widehat{V}_1}\chi\widehat{U}_1,\phi P_1\widehat{Z} + \omega P_3\widehat{Z}) + g_1(\nabla_{\widehat{V}_1}\chi\omega\widehat{U}_1 - \nabla_{\widehat{U}_1}\chi\omega\widehat{V}_1,\widehat{Z})$$
$$\frac{1}{\lambda^2}\{g_2(\nabla^J_{\widehat{U}_1}J_*\chi\widehat{V}_1 - \nabla^J_{\widehat{V}_1}J_*\chi\widehat{U}_1,J_*\chi P_3\widehat{Z})\}.$$

The condition of integrability for $\mathfrak{D}_{\theta_2}$ can be determined in the same way, as shown below:

Theorem 4.2 *Let $J : (\bar{B}_1,\phi,\xi,\eta,g_1) \to (\bar{B}_2,g_2)$ be a $\mathscr{QBSCS}$, where $(\bar{B}_1,\phi,\xi,\eta,g_1)$ a Kenmotsu manifold and $(\bar{B}_2,g_2)$ a RM. Then slant distribution $\mathfrak{D}_{\theta_2}$ is integrable if and only if*

$$-\frac{1}{\lambda^2}\{g_2((\nabla J_*)(\widehat{U}_2,\chi\widehat{V}_2) - (\nabla J_*)(\widehat{V}_2,\chi\widehat{U}_2),J_*\chi P_2\widehat{Z})\}$$
$$= g_1(\mathscr{T}_{\widehat{V}_2}\chi\omega\widehat{U}_2 - \mathscr{T}_{\widehat{U}_2}\chi\omega\widehat{V}_2,\widehat{Z}) + g_1(\mathscr{T}_{\widehat{U}_2}\chi\widehat{V}_2 - \mathscr{T}_{\widehat{V}_2}\chi\widehat{U}_2,\phi P_1\widehat{Z} + \omega P_2\widehat{Z})$$
$$+ \frac{1}{\lambda^2}\{g_2(\nabla^J_{\widehat{U}_2}J_*\chi\widehat{V}_2 - \nabla^J_{\widehat{V}_2}J_*\chi\widehat{U}_2,J_*\chi P_2\widehat{Z})\},$$

for any $\widehat{U}_2,\widehat{V}_2 \in \Gamma(\mathfrak{D}_{\theta_2})$ and $\widehat{Z} \in \Gamma(\mathfrak{D} \oplus \mathfrak{D}_{\theta_1} \oplus <\xi>)$.

Proof. On using equations (2.3), (2.4), (2.6) and (3.18), we have

$$g_1([\widehat{U}_2,\widehat{V}_2],\widehat{Z}) = g_1(\nabla_{\widehat{V}_2}\omega^2\widehat{U}_2,\widehat{Z}) + g_1(\nabla_{\widehat{V}_2}\chi\omega\widehat{U}_2,\widehat{Z}) - g_1(\nabla_{\widehat{U}_2}\omega^2\widehat{V}_2,\widehat{Z})$$
$$- g_1(\nabla_{\widehat{U}_2}\chi\omega\widehat{V}_2,\widehat{Z}) + g_1(\nabla_{\widehat{U}_2}\chi\widehat{V}_2 - \nabla_{\widehat{V}_2}\chi\widehat{U}_2,\phi\widehat{Z}),$$

for any $\widehat{U}_2,\widehat{V}_2 \in \Gamma(\mathfrak{D}_{\theta_2})$ and $\widehat{Z} \in \Gamma(\mathfrak{D} \oplus \mathfrak{D}_{\theta_1} \oplus <\xi>)$. From equation (2.11) and Lemma 3.3, we get

$$sin^2\theta_2 g_1([\widehat{U}_2,\widehat{V}_2],\widehat{Z}) = g_1(\mathscr{T}_{\widehat{V}_2}\chi\omega\widehat{U}_2 - \mathscr{T}_{\widehat{U}_2}\chi\omega\widehat{V}_2,\widehat{Z}) + g_1(\mathscr{T}_{\widehat{U}_2}\chi\widehat{V}_2 - \mathscr{T}_{\widehat{V}_2}\chi\widehat{U}_2,\phi P_1\widehat{Z} + \omega P_2\widehat{Z})$$
$$+ g_1(\mathscr{H}\nabla_{\widehat{U}_2}\chi\widehat{V}_2 - \mathscr{H}\nabla_{\widehat{V}_2}\chi\widehat{U}_2,\chi P_2\widehat{Z}).$$

Since J is $\mathscr{QBSCS}$, using conformality condition with equations (2.7) and (2.16), we finally get

$$\begin{aligned} sin^2\theta_2 g_1([\widehat{U}_2,\widehat{V}_2],\widehat{Z}) = {} & g_1(\mathscr{T}_{\widehat{V}_2}\chi\omega\widehat{U}_2 - \mathscr{T}_{\widehat{U}_2}\chi\omega\widehat{V}_2,\widehat{Z}) + g_1(\mathscr{T}_{\widehat{U}_2}\chi\widehat{V}_2 - \mathscr{T}_{\widehat{V}_2}\chi\widehat{U}_2,\phi P_1\widehat{Z} + \omega P_2\widehat{Z}) \\ & + \frac{1}{\lambda^2}\{g_2((\nabla J_*)(\widehat{U}_2,\chi\widehat{V}_2) - (\nabla J_*)(\widehat{V}_2,\chi\widehat{U}_2), J_*\chi P_2\widehat{Z})\} \\ & + \frac{1}{\lambda^2}\{g_2(\nabla^J_{\widehat{U}_2} J_*\chi\widehat{V}_2 - \nabla^J_{\widehat{V}_2} J_*\chi\widehat{U}_2, J_*\chi P_2\widehat{Z})\}. \end{aligned}$$

This completes the proof of the theorem.

Given that slant distributions and the invariant distribution are mutually orthogonal. This inspired us to look into the prerequisites for the integrability of the invariant distribution $\mathfrak{D}$.

Theorem 4.3 *Let $J : (\bar{B}_1,\phi,\xi,\eta,g_1) \to (\bar{B}_2,g_2)$ be a $\mathscr{QBSCS}$, where $(\bar{B}_1,\phi,\xi,\eta,g_1)$ a Kenmotsu manifold and $(\bar{B}_2,g_2)$ a RM. Then the invariant distribution $\mathfrak{D}$ is integrable if and only if*

$$\begin{aligned} & g_1(\mathscr{T}_{\widehat{U}}\omega P_1\widehat{V} - \mathscr{T}_{\widehat{V}}\omega P_1\widehat{U}, \chi P_2\widehat{Z} + \chi P_3\widehat{W}) \\ & = -g_1(\mathscr{V}\nabla_{\widehat{U}}\omega P_1\widehat{V} - \mathscr{V}\nabla_{\widehat{V}}\omega P_1\widehat{U}, \omega P_2\widehat{Z} + \omega P_3\widehat{Z}), \end{aligned} \tag{4.35}$$

for any $\widehat{U},\widehat{V} \in \Gamma(\mathfrak{D})$ and $\widehat{Z} \in \Gamma(\mathfrak{D}_{\theta_1} \oplus \mathfrak{D}_{\theta_2} \oplus <\xi>)$.

Proof. For all $\widehat{U},\widehat{V} \in \Gamma(\mathfrak{D})$ and $\widehat{Z} \in \Gamma(\mathfrak{D}_{\theta_1} \oplus \mathfrak{D}_{\theta_2} \oplus <\xi>)$ with using equations (2.3), (2.4), (2.10) and decomposition (3.17), we have

$$g_1([\widehat{U},\widehat{V}],\widehat{Z}) = g_1(\nabla_{\widehat{U}}\omega P_1\widehat{V}, \phi P_2\widehat{Z} + \phi P_3\widehat{Z}) - g_1(\nabla_{\widehat{V}}\omega P_1\widehat{U}, \phi P_2\widehat{Z} + \phi P_3\widehat{Z}).$$

On using equation (3.18), we finally have

$$\begin{aligned} g_1([\widehat{U},\widehat{V}],\widehat{Z}) = {} & g_1(\mathscr{T}_{\widehat{U}}\omega P_1\widehat{V} - \mathscr{T}_{\widehat{V}}\omega P_1\widehat{U}, \chi P_2\widehat{Z} + \chi P_3\widehat{Z}) \\ & + g_1(\mathscr{V}\nabla_{\widehat{U}}\omega P_1\widehat{V} - \mathscr{V}\mathscr{A}_{\widehat{V}}\omega P_1\widehat{U}, \omega P_2\widehat{Z} + \omega P_3\widehat{Z}). \end{aligned}$$

This completes the proof of theorem.

After describing the necessary conditions for distributions integrability, we will move on to the necessary and sufficient conditions that must also exist in order for distributions to be totally geodesic. We begin by looking into the prerequisites and criteria for totally geodesic distributions.

Theorem 4.4 *Let $J : (\bar{B}_1,\phi,\xi,\eta,g_1) \to (\bar{B}_2,g_2)$ be $\mathscr{QBSCS}$ from a Kenmotsu manifold onto a Riemannian manifold. $(\bar{B}_2,g_2)$. Then $\mathfrak{D}$ is not defines totally geodesic foliation on $\bar{B}_1$.*

Proof. Taking the vector fields $\widehat{Z},\widehat{V} \in \Gamma(\mathfrak{D})$ and since $\widehat{V}$ and ξ are orthogonal, we have

$$g(\nabla_{\widehat{U}}\widehat{V},\xi) = -g(\widetilde{V},\nabla_{\widehat{U}}\xi)$$

By considering equation (2.8), we get

$$g(\nabla_{\widehat{U}}\widehat{V},\xi) = -g(\widehat{U},\widehat{V}).$$

For $\widehat{U},\widehat{V} \in \Gamma(\mathfrak{D}), g(\widehat{U},\widehat{V}) \neq 0$, that is $g(\nabla_{\widehat{U}}\widehat{V},\xi) \neq 0$. Hence, the distribution is not totally geodesic. Since, the invariant distribution is not defines totally geodesic foliation on $\bar{B}_1$, therefore, we discuss the geometry of leaf of distribution $\mathfrak{D}\oplus <\xi>$.

Theorem 4.5 *Let $J : (\bar{B}_1,\phi,\xi,\eta,g_1) \to (\bar{B}_2,g_2)$ be a $\mathscr{QBSCS}$, where $(\bar{B}_1,\phi,\xi,\eta,g_1)$ a Kenmotsu manifold and $(\bar{B}_2,g_2)$ a RM. Then invariant distribution $\mathfrak{D}\oplus <\xi>$ defines totally geodesic foliation on $\bar{B}_1$ if and only if*

(i) $\lambda^{-2}g_2\{((\nabla J_*)(\widehat{U},\phi\widehat{V}),J_*\chi\widehat{Z})\} = g_1(\mathscr{V}\nabla_{\widehat{U}}\phi\widehat{V},\omega\widehat{Z})$

(ii) $\lambda^{-2}\{g_2((\nabla J_*)(\widehat{U},\phi\widehat{V}),J_*n\widehat{X})\} = g_1(\mathscr{V}\nabla_{\widehat{U}}\phi\widehat{V},t\widehat{X})$,

for any $\widehat{U},\widehat{V} \in \Gamma(\mathfrak{D}\oplus <\xi>)$ and $\widehat{Z} \in \Gamma(\mathfrak{D}_{\theta_1}\oplus\mathfrak{D}_{\theta_2}), \widehat{X} \in \Gamma(KerJ_*)^{\perp}$.

Proof. For any $\widehat{U},\widehat{V} \in \Gamma(\mathfrak{D})$ and $\widehat{Z} \in \Gamma(\mathfrak{D}_{\theta_1}\oplus\mathfrak{D}_{\theta_2})$ with using equations (2.3), (2.4), (2.6) and (3.18), we may write

$$g_1(\nabla_{\widehat{U}}\widehat{V},\widehat{Z}) = g_1(\mathscr{V}\nabla_{\widehat{U}}\phi\widehat{V},\omega\widehat{Z}) + g_1(\mathscr{T}_{\widehat{U}}\phi\widehat{V},\chi\widehat{Z}).$$

On using the conformality of J with equation (2.7) and (2.16), we get

$$g_1(\nabla_{\widehat{U}}\widehat{V},\widehat{Z}) = g_1(\mathscr{V}\nabla_{\widehat{U}}\phi\widehat{V},\omega\widehat{Z}) - \lambda^{-2}g_2((\nabla J_*)(\widehat{U},\phi\widehat{V}),J_*\chi\widehat{Z}).$$

On the other hand, using equations (2.3), (2.4), (2.6) with conformality of J with $\widehat{X} \in \Gamma(KerJ_*)^{\perp}$, we finally have

$$g_1(\nabla_{\widehat{U}}\widehat{V},\widehat{X}) = g_1(\mathscr{V}\nabla_{\widehat{U}}\phi\widehat{V},t\widehat{X}) - \lambda^{-2}g_2((\nabla J_*)(\widehat{U},\phi\widehat{V}),J_*n\widehat{X}),$$

from which we get the desired result.

Theorem 4.6 *Let $J : (\bar{B}_1,\phi,\xi,\eta,g_1) \to (\bar{B}_2,g_2)$ be $\mathscr{QBSCS}$ from a Kenmotsu manifold onto a Riemannian manifold. $(\bar{B}_2,g_2)$. Then $\mathfrak{D}_{\theta_1}$ is not defines totally geodesic foliation on $\bar{B}_1$.*

Proof. Taking the vector fields $\widehat{X},\widehat{Y} \in \Gamma(\mathfrak{D}_{\theta_1})$ and since $\widehat{Y}$ and ξ are orthogonal, we have

$$g(\nabla_{\widehat{X}}\widehat{Y},\xi) = -g(\widetilde{Y},\nabla_{\widehat{X}}\xi)$$

By considering equation (2.8), we get

$$g(\nabla_{\widehat{X}}\widehat{Y},\xi) = -g(\widehat{X},\widehat{Y}).$$

For $\widehat{X},\widehat{Y} \in \Gamma(\mathfrak{D}_{\theta_1}), g(\widehat{X},\widehat{Y}) \neq 0$, that is $g(\nabla_{\widehat{X}}\widehat{Y},\xi) \neq 0$. Hence, the distribution is not totally geodesic.

In same manner, we can examine the geometry of leaves of $\mathfrak{D}_{\theta_1} \oplus \xi$ as follows:

Theorem 4.7 *Let J be a $\mathscr{QBSCS}$ from Kenmotsu manifold $(\bar{B}_1,\phi,\xi,\eta,g_1)$ onto a RM $(\bar{B}_2,g_2)$. Then slant distribution $\mathfrak{D}_{\theta_1} \oplus \xi$ defines totally geodesic foliation on $\bar{B}_1$ if and only if*

$$\begin{aligned}
&\frac{1}{\lambda^2}g_2(\nabla^J_{\widehat{Z}}J_*\chi P_2\widehat{W},J_*\chi P_3\widehat{W}) \\
&= cos^2\theta_1 g_1(\nabla_{\widehat{Z}}P_2\widehat{W},\widehat{U}) - g_1(\mathscr{T}_{\widehat{Z}}\chi\omega P_2\widehat{W},\widehat{U}) + g_1(\mathscr{T}_{\widehat{Z}}\chi\omega P_2\widehat{W},\phi P_1\widehat{U}) \\
&\quad + g_1(\mathscr{T}_{\widehat{Z}}\chi P_2\widehat{W},\omega P_3\widehat{U} - \frac{1}{\lambda^2}g_2((\nabla J_*)(\chi P_2\widehat{W},\widehat{Z}),J_*P_3\widehat{U}) - g_1(\phi\widehat{Z},\widehat{U})\eta(P_2\widehat{W}).
\end{aligned} \tag{4.36}$$

and

$$\begin{aligned}
&\lambda^{-2}\{g_2(\nabla^J_{\widehat{Z}}J_*\chi\omega P_2\widehat{W},J_*\widehat{X})\} + \eta(P_2\widehat{W})g_1(\phi\widehat{Z},\widehat{X}) \\
&= \frac{1}{\lambda^2}g_2((\nabla J_*)(\widehat{Z},\chi\omega P_2\widehat{W}),J_*\widehat{X})) - \frac{1}{\lambda^2}g_2((\nabla J_*)(\widehat{Z},\chi\omega P_2\widehat{W}),J_*n\widehat{X})) \\
&\quad + cos^2\theta_1 g_1(\nabla_{\widehat{Z}}P_2\widehat{W},\widehat{X}) + g_1(\mathscr{T}_{\widehat{Z}}\chi\omega P_2\widehat{W},t\widehat{X}) - g_2(\nabla^J_{\widehat{Z}}J_*\chi\omega P_2\widehat{W},J_*n\widehat{X}),
\end{aligned} \tag{4.37}$$

for any $\widehat{Z},\widehat{W} \in \Gamma(\mathfrak{D}_{\theta_1} \oplus \xi), \widehat{U} \in \Gamma(D \oplus \mathfrak{D}_{\theta_2})$ and $\widehat{X} \in \Gamma(kerJ_*)^{\perp}$.

Proof. By using equations (2.3), (2.6), (2.4) and (3.18), we get

$$g_1(\nabla_{\widehat{Z}}\widehat{W},\widehat{U}) = g_1(\nabla_{\widehat{Z}}\chi P_2\widehat{W},\phi(P_1\widehat{U}+P_3\widehat{U})) - g_1(\phi\nabla_{\widehat{Z}}\omega P_2\widehat{W},\widehat{U}),$$

for $\widehat{Z},\widehat{W} \in \Gamma(\mathfrak{D}_{\theta_1} \oplus \xi)$ and $\widehat{U} \in \Gamma(\mathfrak{D} \oplus \mathfrak{D}_{\theta_2})$. Again using equations (2.3), (2.6), (2.4), (3.18), (2.11) with Lemma 3.2, we may write

$$\begin{aligned}
g_1(\nabla_{\widehat{Z}}\widehat{W},\widehat{U}) &= cos^2\theta_1 g_1(\nabla_{\widehat{Z}}P_2\widehat{W},\widehat{U}) - g_1(\mathscr{T}_{\widehat{Z}}\chi\omega P_2\widehat{W},\widehat{U}) + g_1(\mathscr{T}_{\widehat{Z}}\chi\omega P_2\widehat{W},\phi P_1\widehat{U}) \\
&\quad + g_1(\mathscr{T}_{\widehat{Z}}\chi P_2\widehat{W},\omega P_3\widehat{U}) + g_1(\mathscr{H}\nabla_{\widehat{Z}}\chi P_2\widehat{W},\chi P_3\widehat{U}) - g_1(\phi\widehat{Z},\widehat{U})\eta(P_2\widehat{W}).
\end{aligned}$$

Since, J is conformal, using Lemma 2.1 with equations (2.7) and (2.16), we have

$$\begin{aligned}
g_1(\nabla_{\widehat{Z}}\widehat{W},\widehat{U}) &= cos^2\theta_1 g_1(\nabla_{\widehat{Z}}P_2\widehat{W},\widehat{U}) - g_1(\mathscr{T}_{\widehat{Z}}\chi\omega P_2\widehat{W},\widehat{U}) + g_1(\mathscr{T}_{\widehat{Z}}\chi\omega P_2\widehat{W},\phi P_1\widehat{U}) \\
&\quad + g_1(\mathscr{T}_{\widehat{Z}}\chi P_2\widehat{W},\omega P_3\widehat{U}) - \frac{1}{\lambda^2}g_2(\nabla^J_{\widehat{Z}}J_*\chi P_2\widehat{W},J_*\chi P_3\widehat{W}) \\
&\quad - \frac{1}{\lambda^2}g_2((\nabla J_*)(\chi P_2\widehat{W},\widehat{Z}),J_*P_3\widehat{U}) - g_1(\phi\widehat{Z},\widehat{U})\eta(P_2\widehat{W}).
\end{aligned} \tag{4.38}$$

On the other hand, for $\widehat{Z},\widehat{W}\in\Gamma(\mathfrak{D}_{\theta_1}\oplus\xi)$ and $\widehat{X}\in\Gamma(kerJ_*)^{\perp}$, with using equations (2.3), (2.6), (2.4) and (3.18), we get

$$g_1(\nabla_{\widehat{Z}}\widehat{W},\widehat{X})=g_1(\nabla_{\widehat{Z}}\omega P_2\widehat{W},\phi\widehat{X})+g_1(\nabla_{\widehat{Z}}\chi P_2\widehat{W},\phi\widehat{X})+\eta(P_2\widehat{W})g_1(\phi\widehat{Z},\widehat{X}).$$

From Lemma 3.2 with equations (2.11) and (3.21), the above equation takes the form

$$\begin{aligned}g_1(\nabla_{\widehat{Z}}\widehat{W},\widehat{X})&=cos^2\theta_1 g_1(\nabla_{\widehat{Z}}P_2\widehat{W},\widehat{X})-g_1(\mathscr{H}\nabla_{\widehat{Z}}\chi\omega P_2\widehat{W},\widehat{X})+\eta(P_2\widehat{W})g_1(\phi\widehat{Z},\widehat{X})\\&+g_1(\mathscr{T}_{\widehat{Z}}\chi\omega P_2\widehat{W},t\widehat{X})+g_1(\mathscr{H}\nabla_{\widehat{Z}}\chi\omega P_2\widehat{W},n\widehat{X}).\end{aligned}$$

Since J is conformal and from equations (2.7) and (2.16), we have

$$\begin{aligned}g_1(\nabla_{\widehat{Z}}\widehat{W},\widehat{X})&=cos^2\theta_1 g_1(\nabla_{\widehat{Z}}P_2\widehat{W},\widehat{X})+g_1(\mathscr{T}_{\widehat{Z}}\chi\omega P_2\widehat{W},t\widehat{X})-\eta(P_2\widehat{W})g_1(\phi\widehat{Z},\widehat{X})\\&+\frac{1}{\lambda^2}g_2((\nabla J_*)(\chi\omega P_2\widehat{W},\widehat{Z}),J_*\widehat{X})-\frac{1}{\lambda^2}g_2(\nabla^J_{\widehat{Z}}J_*\chi\omega P_2\widehat{W},J_*\widehat{X})\\&-\frac{1}{\lambda^2}g_2((\nabla J_*)(\chi\omega P_2\widehat{W},\widehat{Z}),J_*n\widehat{X})+\frac{1}{\lambda^2}g_2(\nabla^J_{\widehat{Z}}J_*\chi\omega P_2\widehat{W},J_*n\widehat{X}),\end{aligned}$$

from which we get the result.

Theorem 4.8 *Let $J:(\bar{B}_1,\phi,\xi,\eta,g_1)\to(\bar{B}_2,g_2)$ be $\mathscr{QBSCS}$ from a Kenmotsu manifold onto a Riemannian manifold. $(\bar{B}_2,g_2)$. Then $\mathfrak{D}_{\theta_2}$ is not defines totally geodesic foliation on $\bar{B}_1$.*

Proof. Taking the vector fields $\widehat{X},\widehat{Y}\in\Gamma(\mathfrak{D}_{\theta_2})$ and since $\widehat{Y}$ and ξ are orthogonal, we have

$$g(\nabla_{\widehat{X}}\widehat{Y},\xi)=-g(\widetilde{Y},\nabla_{\widehat{X}}\xi)$$

By considering equation (2.8), we get

$$g(\nabla_{\widehat{X}}\widehat{Y},\xi)=-g(\widehat{X},\widehat{Y}).$$

For $\widehat{X},\widehat{Y}\in\Gamma(\mathfrak{D}_{\theta_2}),g(\widehat{X},\widehat{Y})\neq 0$, that is $g(\nabla_{\widehat{X}}\widehat{Y},\xi)\neq 0$. Hence, the distribution is not totally geodesic.

In the following theorem, we study the necessary and sufficient conditions for slant distribution $\mathfrak{D}_{\theta_2}$ to be totally geodesic.

Theorem 4.9 *Let $J:(\bar{B}_1,\phi,\xi,\eta,g_1)\to(\bar{B}_2,g_2)$ be a $\mathscr{QBSCS}$, where $(\bar{B}_1,\phi,\xi,\eta,g_1)$ a Kenmotsu manifold and $(\bar{B}_2,g_2)$ a RM. Then slant distribution $\mathfrak{D}_{\theta_2}\oplus<\xi>$ defines totally geodesic* *foliation on $\bar{B}_1$ if and only if*

$$\begin{aligned}&\frac{1}{\lambda^2}g_2(\nabla^J_{\widehat{Z}}J_*\chi P_2\widehat{W},J_*\chi P_3\widehat{V})\\&=cos^2\theta_1 g_1(\nabla_{\widehat{Z}}P_2\widehat{W},\widehat{V})-g_1(\mathscr{T}_{\widehat{Z}}\chi\omega P_2\widehat{W},\widehat{V})+g_1(\mathscr{T}_{\widehat{Z}}\chi\omega P_2\widehat{W},\phi P_1\widehat{V})\\&\quad+g_1(\mathscr{T}_{\widehat{Z}}\chi P_2\widehat{W},\omega P_3\widehat{V}-\frac{1}{\lambda^2}g_2((\nabla J_*)(\chi P_2\widehat{W},\widehat{Z}),J_*P_3\widehat{V})-g_1(\phi\widehat{Z},\widehat{V})\eta(P_2\widehat{W}).\end{aligned} \tag{4.39}$$

and

$$\begin{aligned}&\lambda^{-2}\{g_2(\nabla^J_{\widehat{Z}}J_*\chi\omega P_3\widehat{W},J_*\widehat{Y})-g_2(\nabla^J_{\widehat{Z}}J_*\chi\omega P_3\widehat{W},J_*n\widehat{Y})\}+\eta(P_2\widehat{W})g_1(\phi\widehat{Z},\widehat{Y})\\&=\frac{1}{\lambda^2}g_2((\nabla J_*)(\widehat{Z},\chi\omega P_3\widehat{W}),J_*\widehat{Y})-\frac{1}{\lambda^2}g_2((\nabla J_*)(\widehat{Z},\chi\omega P_3\widehat{W}),J_*n\widehat{Y})\\&\quad+cos^2\theta_2 g_1(\nabla_{\widehat{Z}}P_3\widehat{W},\widehat{Y})+g_1(\mathscr{T}_{\widehat{Z}}\chi\omega P_3\widehat{W},t\widehat{Y})+\eta(\widehat{W})g_1(\widehat{Z},t\widehat{Y}),\end{aligned} \tag{4.40}$$

for any $\widehat{Z},\widehat{W}\in\Gamma(D_{\theta_2}\oplus<\xi>),\widehat{V}\in\Gamma(D\oplus D_{\theta_1})$ and $\widehat{Y}\in\Gamma(kerJ_*)^{\perp}$.

Proof. The proof of above theorem is similar to the proof of Theorem 4.7.

Since, J is $\mathscr{QBSCS}$, having $(kerJ_*)$ and $(kerJ_*)^{\perp}$ are vertical and horizontal distributions, respectively. We now investigate the conditions under which distributions define totally geodesic foliation on $\bar{B}_1$. In terms of vertical distribution's total geodesicness, we have

Theorem 4.10 *Let $J:(\bar{B}_1,\phi,\xi,\eta,g_1)\to(\bar{B}_2,g_2)$ be a $\mathscr{QB}\vec{\mathscr{S}}\mathscr{CS}$, where $(\bar{B}_1,\phi,\xi,\eta,g_1)$ a Kenmotsu manifold and $(\bar{B}_2,g_2)$ a RM. Then $kerJ_*$ defines totally geodesic foliation on $\bar{B}_1$ if and only if*

$$\begin{aligned}&\frac{1}{\lambda^2}\{g_2(\nabla^J_{\widehat{U}}J_*\chi\omega P_2\widehat{V}+\nabla^J_{\widehat{U}}J_*\chi\omega P_3\widehat{V},J_*\widehat{X})\}\\&=g_1(\mathscr{T}_{\widehat{U}}P_1\widehat{V}+cos^2\theta_1\mathscr{T}_{\widehat{U}}P_2\widehat{V}+cos^2\theta_2\mathscr{T}_{\widehat{U}}P_3\widehat{V},\widehat{X})+g_1(\mathscr{T}_{\widehat{U}}\chi\widehat{V},t\widehat{X})\\&\quad+\frac{1}{\lambda^2}\{g_2((\nabla J_*)(\widehat{U},\chi\omega P_2\widehat{V})-(\nabla J_*)(\widehat{U},\chi\omega P_3\widehat{V}),J_*\widehat{X})\}\\&\quad+\frac{1}{\lambda^2}\{g_2(\nabla^J_{\widehat{U}}J_*\chi\widehat{V}-(\nabla J_*)(\widehat{U},\chi\widehat{V}),J_*n\widehat{X})\}.\end{aligned} \tag{4.41}$$

for any $\widehat{U},\widehat{V}\in\Gamma(kerJ_*)$ and $\widehat{X}\in\Gamma(kerJ_*)^{\perp}$.

Proof. For any $\widehat{U},\widehat{V}\in\Gamma(kerJ_*)$ and $\widehat{X}\in\Gamma(kerJ_*)^{\perp}$ with using equations (2.3), (2.6), (2.4) with decomposition (3.17), we get

$$g_1(\nabla_{\widehat{U}}\widehat{V},\widehat{X})=g_1(\nabla_{\widehat{U}}\phi P_1\widehat{V},\phi\widehat{X})+g_1(\nabla_{\widehat{U}}\phi P_2\widehat{V},\phi\widehat{X})+g_1(\nabla_{\widehat{U}}\phi P_3\widehat{V},\phi\widehat{X}).$$

On using equation (3.18) with Lemma 3.2 and Lemma 3.3, we have

$$\begin{aligned} g_1(\nabla_{\widehat{U}}\widehat{V},\widehat{X}) = {} & g_1(\nabla_{\widehat{U}}P_1\widehat{V},\widehat{X}) + cos^2\theta_1 g_1(\nabla_{\widehat{U}}P_2\widehat{V},\widehat{X}) + cos^2\theta_2 g_1(\nabla_{\widehat{U}}P_3\widehat{V},\widehat{X}) \\ & + g_1(\nabla_{\widehat{U}}\chi P_2\widehat{V},\phi\widehat{X}) - g_1(\nabla_{\widehat{U}}\chi\omega P_2\widehat{V},\widehat{X}) - g_1(\nabla_{\widehat{U}}\chi\omega P_3\widehat{V},\widehat{X}) \\ & + g_1(\nabla_{\widehat{U}}\chi P_3\widehat{V},\phi\widehat{X}) - \eta(P_2\widehat{V})g_1(\chi\widehat{U},\widehat{X}). \end{aligned}$$

From equations (2.10), (2.11) and (3.21), we may yields

$$\begin{aligned} g_1(\nabla_{\widehat{U}}\widehat{V},\widehat{X}) = {} & g_1(\mathscr{T}_{\widehat{U}}P_1\widehat{V} + cos^2\theta_1\mathscr{T}_{\widehat{U}}P_2\widehat{V} + cos^2\theta_2\mathscr{T}_{\widehat{U}}P_3\widehat{V},\widehat{X}) \\ & - g_1(\mathscr{H}\nabla_{\widehat{U}}\chi\omega P_2\widehat{V} + \mathscr{H}\nabla_{\widehat{U}}\chi\omega P_3\widehat{V},\widehat{X}) + g_1(\mathscr{T}_{\widehat{U}}\chi P_2\widehat{V} + \mathscr{T}_{\widehat{U}}\chi P_3\widehat{V},t\widehat{X}) \\ & + g_1(\mathscr{H}\nabla_{\widehat{U}}\chi P_2\widehat{V} + \mathscr{H}\nabla_{\widehat{U}}\chi P_3\widehat{V},n\widehat{X}) - \eta(P_2\widehat{V})g_1(\chi\widehat{U},\widehat{X}). \end{aligned}$$

From decomposition (3.17), the above equation takes the form

$$\begin{aligned} g_1(\nabla_{\widehat{U}}\widehat{V},\widehat{X}) = {} & g_1(\mathscr{T}_{\widehat{U}}P_1\widehat{V} + cos^2\theta_1\mathscr{T}_{\widehat{U}}P_2\widehat{V} + cos^2\theta_2\mathscr{T}_{\widehat{U}}P_3\widehat{V},\widehat{X}) + g_1(\mathscr{T}_{\widehat{U}}\chi\widehat{V},t\widehat{X}) \\ & - g_1(\mathscr{H}\nabla_{\widehat{U}}\chi\omega P_2\widehat{V} + \mathscr{H}\nabla_{\widehat{U}}\chi\omega P_3\widehat{V},\widehat{X}) + g_1(\mathscr{H}\nabla_{\widehat{U}}\chi\widehat{V},n\widehat{X}) \\ & - \eta(P_2\widehat{V})g_1(\chi\widehat{U},\widehat{X}). \end{aligned}$$

Using the conformality of J with equations (2.7) and (2.16), we have

$$\begin{aligned} g_1(\nabla_{\widehat{U}}\widehat{V},\widehat{X}) = {} & g_1(\mathscr{T}_{\widehat{U}}P_1\widehat{V} + cos^2\theta_1\mathscr{T}_{\widehat{U}}P_2\widehat{V} + cos^2\theta_2\mathscr{T}_{\widehat{U}}P_3\widehat{V},\widehat{X}) + g_1(\mathscr{T}_{\widehat{U}}\chi\widehat{V},t\widehat{X}) \\ & + \frac{1}{\lambda^2}\{g_2((\nabla J_*)(\widehat{U},\chi\omega P_2\widehat{V}) - (\nabla J_*)(\widehat{U},\chi\omega P_3\widehat{V}),J_*\widehat{X})\} \\ & - \frac{1}{\lambda^2}\{g_2(\nabla^J_{\widehat{U}}J_*\chi\omega P_2\widehat{V} + \nabla^J_{\widehat{U}}J_*\chi\omega P_3\widehat{V},J_*\widehat{X})\} \\ & + \frac{1}{\lambda^2}\{g_2(\nabla^J_{\widehat{U}}J_*\chi\widehat{V} - (\nabla J_*)(\widehat{U},\chi\widehat{V}),J_*n\widehat{X})\} \\ & - \eta(P_2\widehat{V})g_1(\chi\widehat{U},\widehat{X}). \end{aligned}$$

This completes the proof of the theorem.

Now we can discuss the geometry of the horizontal distribution's leaves. The necessary and sufficient conditions under which horizontal distribution totally geodesic foliation on $\bar{B}_1$ are presented in the following theorem.

Theorem 4.11 *Let J be a $\mathscr{QBSCS}$ from Kenmotsu manifold $(\bar{B}_1,\phi,\xi,\eta,g_1)$ onto a RM $(\bar{B}_2,g_2)$,.*

Then $(kerJ_*)^{\perp}$ *defines totally geodesic foliation on* $\bar{B}_1$ *if and only if*

$$\begin{aligned}
&-\frac{1}{\lambda^2}g_2(\nabla^J_{\widehat{X}}J_*n\widehat{Y},J_*\chi Z)+\frac{1}{\lambda^2}g_2(\nabla^J_{\widehat{X}}J_*\widehat{Y},J_*\chi\omega P_2Z)\\
&=\cos^2\theta_1 g_1(\mathscr{A}_{\widehat{X}}\widehat{Y},P_2Z)+\cos^2\theta_2 g_1(\mathscr{A}_{\widehat{X}}\widehat{Y},P_3Z)\\
&\quad+g_1(\mathscr{V}\nabla_{\widehat{X}}t\widehat{Y},\omega P_1Z)+g_1(\mathscr{A}_{\widehat{X}}n\widehat{Y},\omega P_1Z)+g_1(\mathscr{A}_{\widehat{X}}t\widehat{Y},\chi Z)\\
&\quad+\frac{1}{\lambda^2}g_2(\widehat{X}(\ln\lambda)J_*n\widehat{Y}+n\widehat{Y}(\ln\lambda)J_*\widehat{X}-g_1(\widehat{X},n\widehat{Y})J_*(G\ln\lambda),J_*\chi Z)\\
&\quad+\frac{1}{\lambda^2}g_2(\widehat{X}(\ln\lambda)J_*\widehat{Y}+\widehat{Y}(\ln\lambda)J_*\widehat{X}-g_1(\widehat{X},\widehat{Y})J_*(G\ln\lambda),J_*\chi\omega P_2Z)\\
&\quad+\frac{1}{\lambda^2}g_2(\widehat{X}(\ln\lambda)J_*\widehat{Y}+\widehat{Y}(\ln\lambda)J_*\widehat{X}-g_1(\widehat{X},\widehat{Y})J_*(G\ln\lambda),J_*\chi\omega P_3Z)\\
&\quad+\frac{1}{\lambda^2}g_2(\nabla^J_{\widehat{X}}J_*\widehat{Y},J_*\chi\omega P_3Z)
\end{aligned}\tag{4.42}$$

for any $\widehat{X},\widehat{Y}\in\Gamma(kerJ_*)^{\perp}$ *and* $\widehat{Z}\in\Gamma(kerJ_*)$.

Proof. For any $\widehat{X},\widehat{Y}\in\Gamma(kerJ_*)^{\perp}$ and $\widehat{Z}\in\Gamma(kerJ_*)$ with using equations (2.3), (2.4), (2.6) with decomposition (3.17), we get

$$g_1(\nabla_{\widehat{X}}\widehat{Y},\widehat{Z})=g_1(\nabla_{\widehat{X}}\phi\widehat{Y},\phi P_1\widehat{Z})+g_1(\nabla_{\widehat{X}}\phi\widehat{Y},\phi P_2\widehat{Z})+g_1(\nabla_{\widehat{X}}\phi\widehat{Y},\phi P_3\widehat{Z}).$$

From equations (3.18) and (2.12) with Lemma 3.2, we have

$$\begin{aligned}
&g_1(\nabla_{\widehat{X}}\widehat{Y},\widehat{Z})\\
&=g_1(\mathscr{V}\nabla_{\widehat{X}}t\widehat{Y},\omega P_1Z)+g_1(\mathscr{A}_{\widehat{X}}n\widehat{Y},\omega P_1Z)+g_1(\phi\nabla_{\widehat{X}}\phi\widehat{Y},\phi\omega P_2Z)\\
&\quad+g_1(\nabla_{\widehat{X}}t\widehat{Y},\chi P_2Z)+g_1(\nabla_{\widehat{X}}n\widehat{Y},\chi P_2Z)+g_1(\phi\nabla_{\widehat{X}}\phi\widehat{Y},\phi\omega P_3Z)\\
&\quad+g_1(\nabla_{\widehat{X}}t\widehat{Y},\chi P_3Z)+g_1(\nabla_{\widehat{X}}n\widehat{Y},\chi P_3Z)+g_1(\mathscr{H}\nabla_{\widehat{X}}n\widehat{Y}+\mathscr{A}_{\widehat{X}}t\widehat{Y},\chi P_2\widehat{Z}).
\end{aligned}$$

Since $\chi P_2Z+\chi P_3Z=\chi Z$ and with using the equations (3.18) and (2.13), we get

$$\begin{aligned}
&g_1(\nabla_{\widehat{X}}\widehat{Y},\widehat{Z})\\
&=g_1(\mathscr{V}\nabla_{\widehat{X}}t\widehat{Y},\omega P_1Z)+g_1(\mathscr{A}_{\widehat{X}}n\widehat{Y},\omega P_1Z)+g_1(\mathscr{A}_{\widehat{X}}t\widehat{Y},\chi Z)\\
&\quad+g_1(\mathscr{H}\nabla_{\widehat{X}}n\widehat{Y},\chi Z)-g_1(\mathscr{H}\nabla_{\widehat{X}}\widehat{Y},\chi\omega P_2Z)-g_1(\mathscr{H}\nabla_{\widehat{X}}\widehat{Y},\chi\omega P_3Z)\\
&\quad+\cos^2\theta_1\{g_1(\mathscr{A}_{\widehat{X}}\widehat{Y},P_2Z)\}+\cos^2\theta_2\{g_1(\mathscr{A}_{\widehat{X}}\widehat{Y},P_3Z)\}+g_1(\mathscr{H}\nabla_{\widehat{X}}n\widehat{Y}+\mathscr{A}_{\widehat{X}}t\widehat{Y},\chi P_2\widehat{Z}).
\end{aligned}$$

From formula (2.7) and (2.16), which yields that

$$
\begin{aligned}
& g_1(\nabla_{\widehat{X}}\widehat{Y},\widehat{Z}) \\
& = g_1(\mathscr{V}\nabla_{\widehat{X}}t\widehat{Y},\omega P_1 Z) + g_1(\mathscr{A}_{\widehat{X}}n\widehat{Y},\omega P_1 Z) + g_1(\mathscr{A}_{\widehat{X}}t\widehat{Y},\chi Z) \\
& \quad + \frac{1}{\lambda^2}g_2(\nabla^J_{\widehat{X}}J_*n\widehat{Y},J_*\chi Z) - \frac{1}{\lambda^2}g_2((\nabla J_*)(\widehat{X},n\widehat{Y}),J_*\chi Z) \\
& \quad - \frac{1}{\lambda^2}g_2(\nabla^J_{\widehat{X}}J_*\widehat{Y},J_*\chi\omega P_2 Z) + \frac{1}{\lambda^2}g_2((\nabla J_*)(\widehat{X},\widehat{Y}),J_*\chi\omega P_2 Z) \\
& \quad - \frac{1}{\lambda^2}g_2(\nabla^J_{\widehat{X}}J_*\widehat{Y},J_*\chi\omega P_3 Z) + \frac{1}{\lambda^2}g_2((\nabla J_*)(\widehat{X},\widehat{Y}),J_*\chi\omega P_3 Z) \\
& \quad + \cos^2\theta_1\{g_1(\mathscr{A}_{\widehat{X}}\widehat{Y},P_2 Z)\} + \cos^2\theta_2\{g_1(\mathscr{A}_{\widehat{X}}\widehat{Y},P_3 Z)\} + g_1(\mathscr{H}\nabla_{\widehat{X}}n\widehat{Y} + \mathscr{A}_{\widehat{X}}t\widehat{Y},\chi P_2\widehat{Z}).
\end{aligned}
$$

Since J is conformal submersion, then we finally get

$$
\begin{aligned}
& g_1(\nabla_{\widehat{X}}\widehat{Y},\widehat{Z}) \\
& = \cos^2\theta_1 g_1(\mathscr{A}_{\widehat{X}}\widehat{Y},P_2 Z) + \cos^2\theta_2 g_1(\mathscr{A}_{\widehat{X}}\widehat{Y},P_3 Z) + g_1(\mathscr{H}\nabla_{\widehat{X}}n\widehat{Y} + \mathscr{A}_{\widehat{X}}t\widehat{Y},\chi P_2\widehat{Z}) \\
& \quad + g_1(\mathscr{V}\nabla_{\widehat{X}}t\widehat{Y} + \mathscr{A}_{\widehat{X}}n\widehat{Y},\omega P_1 Z) + g_1(\mathscr{A}_{\widehat{X}}t\widehat{Y},\chi Z) + \frac{1}{\lambda^2}g_2(\nabla^J_{\widehat{X}}J_*\widehat{Y},J_*\chi\omega P_3 Z) \\
& \quad + \frac{1}{\lambda^2}g_2(\widehat{X}(\ln\lambda)J_*n\widehat{Y} + n\widehat{Y}(\ln\lambda)J_*\widehat{X} - g_1(\widehat{X},n\widehat{Y})J_*(G\ln\lambda),J_*\chi Z) \\
& \quad + \frac{1}{\lambda^2}g_2(\widehat{X}(\ln\lambda)J_*\widehat{Y} + \widehat{Y}(\ln\lambda)J_*\widehat{X} - g_1(\widehat{X},\widehat{Y})J_*(G\ln\lambda),J_*\chi\omega P_2 Z) \\
& \quad + \frac{1}{\lambda^2}g_2(\widehat{X}(\ln\lambda)J_*\widehat{Y} + \widehat{Y}(\ln\lambda)J_*\widehat{X} - g_1(\widehat{X},\widehat{Y})J_*(G\ln\lambda),J_*\chi\omega P_3 Z) \\
& \quad + \frac{1}{\lambda^2}g_2(\nabla^J_{\widehat{X}}J_*n\widehat{Y},J_*\chi Z) - \frac{1}{\lambda^2}g_2(\nabla^J_{\widehat{X}}J_*\widehat{Y},J_*\chi\omega P_2 Z).
\end{aligned}
$$

This completes the proof of theorem.

5 ϕ-Pluriharmonicity of Quasi bi-slant Conformal $\xi^\perp$-Submersion

In [23], Y. Ohnita constructed J-pluriharminicity from a almost hermitian manifold. We expand the idea of ϕ-pluriharmonicity to almost contact metric manifolds in this section.

Let J be a $\mathscr{QBSCS}$ from Kenmotsu manifold $(\bar{B}_1,\phi,\xi,\eta,g_1)$ onto a RM $(\bar{B}_2,g_2)$ with slant angles θ_1 and θ_2. Then $\mathscr{QBSC}$ submersion is ϕ-pluriharmonic, $\mathfrak{D}$-ϕ-pluriharmonic, $\mathfrak{D}^{\theta_i}$-ϕ-pluriharmonic, $(\mathfrak{D}-\mathfrak{D}^{\theta_i})$-$\phi$ pluriharmonic (where $i=1,2$), $kerJ_*$-ϕ-pluriharmonic, $(kerJ_*)^\perp$-ϕ-pluriharmonic and $((kerJ_*)^\perp - kerJ_*)$-$\phi$-pluriharmonic if

$$
(\nabla J_*)(\widehat{U},\widehat{V}) + (\nabla J_*)(\phi\widehat{U},\phi\widehat{V}) = 0, \tag{5.43}
$$

for any $\widehat{U},\widehat{V}\in\Gamma(\mathfrak{D})$, for any $\widehat{U},\widehat{V}\in\Gamma(\mathfrak{D}^{\theta_i})$, for any $\widehat{U}\in\Gamma(\mathfrak{D}),\widehat{V}\in\Gamma(\mathfrak{D}^{\theta_i})$ (where $i=1,2$), for any $\widehat{U},\widehat{V}\in\Gamma(kerJ_*)$, for any $\widehat{U},\widehat{V}\in\Gamma(kerJ_*)^{\perp}$ and for any $\widehat{U}\in\Gamma(kerJ_*)^{\perp},\widehat{V}\in\Gamma(kerJ_*)$.

Theorem 5.1 *Let J be a $\mathscr{QBSCS}$ from Kenmotsu manifold $(\bar{B}_1,\phi,\xi,\eta,g_1)$ onto a RM $(\bar{B}_2,g_2)$ with slant angles θ_1 and θ_2. Suppose that J is $\mathfrak{D}_{\theta_1}$-ϕ-pluriharmonic. Then $\mathfrak{D}_{\theta_1}$ defines totally geodesic foliation $\bar{B}_1$ if and only if*

$$\begin{aligned}
&J_*(\chi\mathscr{T}_{\omega\widehat{U}}\chi\omega\widehat{V}+n\mathscr{H}\nabla_{\omega\widehat{U}}\chi\omega\widehat{V})-J_*(\mathscr{A}_{\chi\widehat{U}}\omega\widehat{V}+\mathscr{H}\nabla_{\omega\widehat{U}}\chi\widehat{V})\\
&=\cos^2\theta_1 J_*(n\mathscr{T}_{\omega\widehat{U}}\widehat{V}+\chi\mathscr{V}\nabla_{\omega\widehat{U}}\widehat{V})+\nabla^J_{\omega\widehat{U}}J_*\phi\widehat{V}\\
&\quad-\chi\widehat{U}(\ln\lambda)J_*\chi\widehat{V}-\chi\widehat{V}(\ln\lambda)J_*\chi\widehat{U}+g_1(\chi\widehat{U},\chi\widehat{V})J_*(G\ln\lambda)
\end{aligned}$$

for any $\widehat{U},\widehat{V}\in\Gamma(\mathfrak{D}_{\theta_1})$.

Proof. For any $\widehat{U},\widehat{V}\in\Gamma(\mathfrak{D}_{\theta_1})$ and since, J is $\mathfrak{D}_{\theta_1}$-ϕ-pluriharmonic, then by using equation (2.10) and (2.16), we have

$$\begin{aligned}
0=&(\nabla J_*)(\widehat{U},\widehat{V})+(\nabla J_*)(\phi\widehat{U},\phi\widehat{V})\\
J_*(\nabla_{\widehat{U}}\widehat{V})=&-J_*(\nabla_{\phi\widehat{U}}\phi\widehat{V})+\nabla^J_{\phi\widehat{U}}J_*(\phi\widehat{V})\\
=&\ -J_*(\mathscr{A}_{\chi\widehat{U}}\omega\widehat{V}+\mathscr{V}\nabla_{\chi\widehat{U}}\omega\widehat{V}+\mathscr{T}_{\omega\widehat{U}}\chi\widehat{V}+\mathscr{H}\nabla_{\omega\widehat{U}}\chi\widehat{V})\\
&+(\nabla J_*)(\chi\widehat{U},\chi\widehat{V})-\nabla^J_{\chi\widehat{U}}J_*\chi\widehat{V}+\nabla^J_{\phi\widehat{U}}J_*\phi\widehat{V}\\
&+J_*(\phi\nabla_{\omega\widehat{U}}\phi\omega\widehat{V})
\end{aligned}$$

On using equations (3.18), (3.21) with Lemma 2.1 and Lemma 3.2, the above equation finally takes the form

$$\begin{aligned}
J_*(\nabla_{\widehat{U}}V)=&-\cos^2\theta_1 J_*(P\mathscr{T}_{\omega\widehat{U}}\widehat{V}+n\mathscr{T}_{\omega\widehat{U}}\widehat{V}+\omega\mathscr{V}\nabla_{\omega\widehat{U}}\widehat{V}+\chi\mathscr{V}\nabla_{\omega\widehat{U}}\widehat{V})\\
&+J_*(\omega\mathscr{T}_{\omega\widehat{U}}\chi\omega\widehat{V}+\chi\mathscr{T}_{\omega\widehat{U}}\chi\omega\widehat{V}+P\mathscr{H}\nabla_{\omega\widehat{U}}\chi\omega\widehat{V}+n\mathscr{H}\nabla_{\omega\widehat{U}}\chi\omega\widehat{V})\\
&-J_*(\mathscr{A}_{\chi\widehat{U}}\omega\widehat{V}+\mathscr{V}\nabla_{\chi\widehat{U}}\omega\widehat{V}+\mathscr{T}_{\omega\widehat{U}}\chi\widehat{V}+\mathscr{H}\nabla_{\omega\widehat{U}}\chi\widehat{V})\\
&+\chi\widehat{U}(\ln\lambda)J_*\chi\widehat{V}+\chi\widehat{V}(\ln\lambda)J_*\chi\widehat{U}-g_M(\chi\widehat{U},\chi\widehat{V})J_*(grad\ln\lambda)\\
&-\nabla^J_{\chi\widehat{U}}J_*\chi\widehat{V}+\nabla^J_{\phi\widehat{U}}J_*\phi\widehat{V}.
\end{aligned}$$

from which we get the desired result.

Theorem 5.2 *Let $\vec{f}$ be a $\mathscr{QBSCS}$ from Kenmotsu manifold $(\bar{B}_1,\phi,\xi,\eta,g_1)$ onto a RM $(\bar{B}_2,g_2)$ with slant angles θ_1 and θ_2. Suppose that J is $\mathfrak{D}_{\theta_2}$-ϕ-pluriharmonic. Then $\mathfrak{D}_{\theta_2}$ defines totally geodesic foliation $\bar{B}_1$ if and only if*

$$\begin{aligned}
&J_*(\chi\mathscr{T}_{\omega\widehat{Z}}\chi\omega\widehat{W}+n\mathscr{H}\nabla_{\omega\widehat{Z}}\chi\omega\widehat{W})-J_*(\mathscr{A}_{\chi\widehat{Z}}\omega\widehat{W}+\mathscr{H}\nabla_{\omega\widehat{Z}}\chi\widehat{W})\\
&=\cos^2\theta_2 J_*(n\mathscr{T}_{\omega\widehat{Z}}\widehat{W}+\chi\widehat{\mathscr{W}}\nabla_{\omega\widehat{Z}}\widehat{W})+\nabla^J_{\omega\widehat{Z}}J_*\phi\widehat{W}\\
&\quad-\chi\widehat{Z}(\ln\lambda)J_*\chi\widehat{W}-\chi\widehat{W}(\ln\lambda)J_*\chi\widehat{Z}+g_M(\chi\widehat{Z},\chi\widehat{W})J_*(grad\ln\lambda)
\end{aligned}$$

for any $\widehat{Z},\widehat{W}\in\Gamma(\mathfrak{D}_{\theta_2})$.

Proof. Theorem 5.1 proof is the same to this theorem's proof.

Theorem 5.3 *Let* $\vec{f}$ *be a* $\mathscr{QBSCS}$ *from Kenmotsu manifold* $(\bar{B}_1,\phi,\xi,\eta,g_1)$ *onto a RM* $(\bar{B}_2,g_2)$ *with slant angles* θ_1 *and* θ_2. *Suppose that* J *is* $((kerJ_*)^{\perp}-kerJ_*)$-$\phi$*-pluriharmonic. Then the following assertion are equivalent.*

(i) The horizontal distribution $(kerJ_*)^{\perp}$ *defines totally geodesic foliation on* $\bar{B}_1$.

(ii)
$$\begin{aligned}&(cos^2\theta_1+cos^2\theta_2)J_*\{n\mathscr{T}_{t\widehat{X}}\omega P_1\widehat{U}+\chi\mathscr{V}\nabla_{t\widehat{X}}\omega P_1\widehat{U}+n\mathscr{A}_{n\widehat{X}}\omega P_1\widehat{U}+\chi\mathscr{V}\nabla_{n\widehat{X}}\omega P_1\widehat{U}\}\\&+J_*\{\chi\mathscr{A}_{n\widehat{X}}\chi\omega P_2\widehat{U}+\chi\mathscr{A}_{n\widehat{X}}\chi\omega P_3\widehat{U}-\mathscr{H}\nabla_{t\widehat{X}}\chi\widehat{U}\}+\nabla^J_{n\widehat{X}}J_*\chi\omega P_2\widehat{U}+\nabla^J_{n\widehat{X}}J_*\chi\omega P_3\widehat{U}\\&=J_*\{n\mathscr{T}_{t\widehat{X}}\omega P_1\widehat{U}+\chi\mathscr{V}\nabla_{t\widehat{X}}\omega P_1\widehat{U}+n\mathscr{A}_{n\widehat{X}}\omega P_1\widehat{U}+\chi\mathscr{H}\nabla_{n\widehat{X}}\omega P_1\widehat{U}\}\\&-J_*\{\chi\mathscr{T}_{t\widehat{X}}\chi\omega P_2\widehat{U}+n\mathscr{H}\nabla_{t\widehat{X}}\chi\omega P_2\widehat{U}+\chi\mathscr{T}_{t\widehat{X}}\chi\omega P_3\widehat{U}+n\mathscr{H}\nabla_{t\widehat{X}}\chi\omega P_3\widehat{U}\}\\&+n\widehat{X}(\ln\lambda)J_*\chi\omega P_2\widehat{U}+\chi\omega P_2\widehat{U}(\ln\lambda)J_*n\widehat{X}-g_1(n\widehat{X},\chi\omega P_2\widehat{U})J_*(grad\ln\lambda)\\&+n\widehat{X}(\ln\lambda)J_*\chi\omega P_3\widehat{U}+\chi\omega P_3\widehat{U}(\ln\lambda)J_*n\widehat{X}-g_1(n\widehat{X},\chi\omega P_3\widehat{U})J_*(grad\ln\lambda)\\&+J_*(\nabla_{\widehat{X}}\widehat{U})+\nabla^J_{\phi\widehat{X}}J_*\chi\widehat{U}+g_1(P\widehat{X},\omega\widehat{U})J_*\xi.\end{aligned}$$

for any $\widehat{X}\in\Gamma(kerJ_*)^{\perp}$ *and* $\widehat{U}\in\Gamma(kerJ_*)$

Proof. For any $\widehat{X}\in\Gamma(kerJ_*)^{\perp}$ and $\widehat{U}\in\Gamma(kerJ_*)$, since J is $((kerJ_*)^{\perp}-kerJ_*)$-$\phi$-pluriharmonic, then by using (2.16), (3.18) and (3.21), we get

$$J_*(\nabla_{n\widehat{X}}\chi\widehat{U})=-J_*(\nabla_{t\widehat{X}}\omega\widehat{U}+\nabla_{t\widehat{X}}\chi\widehat{U}+\nabla_{n\widehat{X}}\omega\widehat{U})+J_*(\nabla_{\widehat{X}}\widehat{U})+\nabla^J_{\phi\widehat{X}}J_*\chi\widehat{U}.$$

Taking account the fact from (2.2) and (2.11), we have

$$\begin{aligned}J_*(\nabla_{n\widehat{X}}\chi\widehat{U})=&-J_*(\mathscr{T}_{t\widehat{X}}\chi\widehat{U}+\mathscr{H}\nabla_{t\widehat{X}}\chi\widehat{U})+J_*(\nabla_{\widehat{X}}\widehat{U})+\nabla^J_{\phi\widehat{X}}J_*\chi\widehat{U}\\&+J_*\{\phi\nabla_{t\widehat{X}}\phi\omega\widehat{U}\}+J_*\{\phi\nabla_{n\widehat{X}}\phi\omega\widehat{U}\}.\end{aligned}$$

Now on using decomposition (3.17), Lemma 3.2, Lemma 3.3 with equations (3.18), we may yields

$$\begin{aligned}J_*(\nabla_{n\widehat{X}}\chi\widehat{U})=&J_*\{\phi\nabla_{t\widehat{X}}\omega P_1\widehat{U}-\cos^2\theta_1\phi\nabla_{t\widehat{X}}\omega\widehat{U}-\cos^2\theta_2\phi\nabla_{t\widehat{X}}\omega\widehat{U}\\&+J_*\{\phi\nabla_{n\widehat{X}}\omega P_1\widehat{U}-\cos^2\theta_1\phi\nabla_{n\widehat{X}}\omega\widehat{U}-\cos^2\theta_2\phi\nabla_{n\widehat{X}}\omega\widehat{U}\\&+J_*\{\phi\nabla_{t\widehat{X}}\chi\omega P_2\widehat{U}+\phi\nabla_{t\widehat{X}}\chi\omega P_3\widehat{U}+\phi\nabla_{n\widehat{X}}\chi\omega P_2\widehat{U}+\phi\nabla_{n\widehat{X}}\chi\omega P_3\widehat{U}\}\\&-J_*(\mathscr{H}\nabla_{t\widehat{X}}\chi\widehat{U})+J_*(\nabla_{\widehat{X}}\widehat{U})+\nabla^J_{\phi\widehat{X}}J_*\chi\widehat{U}.\end{aligned}$$

From equations (2.10)-(2.13) and after simple calculation, we may write

$$\begin{aligned}J_*(\nabla_{n\widehat{X}}\chi\widehat{U}) = &-(cos^2\theta_1+cos^2\theta_2)J_*\{n\mathscr{T}_{t\widehat{X}}\omega P_1\widehat{U}+\chi\mathscr{V}\nabla_{t\widehat{X}}\omega P_1\widehat{U}+n\mathscr{A}_{n\widehat{X}}\omega P_1\widehat{U}\\ &+\chi\mathscr{V}\nabla_{n\widehat{X}}\omega P_1\widehat{U}\}-J_*\{\chi\mathscr{A}_{n\widehat{X}}\chi\omega P_2\widehat{U}+\chi\mathscr{A}_{n\widehat{X}}\chi\omega P_3\widehat{U}-\mathscr{H}\nabla_{t\widehat{X}}\chi\widehat{U}\}\\ &+J_*\{n\mathscr{T}_{t\widehat{X}}\omega P_1\widehat{U}+\chi\mathscr{V}\nabla_{t\widehat{X}}\omega P_1\widehat{U}+n\mathscr{A}_{n\widehat{X}}\omega P_1\widehat{U}+\chi\mathscr{H}\nabla_{n\widehat{X}}\omega P_1\widehat{U}\}\\ &-J_*\{\chi\mathscr{T}_{t\widehat{X}}\chi\omega P_2\widehat{U}+n\mathscr{H}\nabla_{t\widehat{X}}\chi\omega P_2\widehat{U}+\chi\mathscr{T}_{t\widehat{X}}\chi\omega P_3\widehat{U}+n\mathscr{H}\nabla_{t\widehat{X}}\chi\omega P_3\widehat{U}\}\\ &-J_*(n\mathscr{H}\nabla_{n\widehat{X}}\chi\omega P_2\widehat{U}+n\mathscr{H}\nabla_{n\widehat{X}}\chi\omega P_2\widehat{U})+J_*(\nabla_{\widehat{X}}\widehat{U})+\nabla^J_{\phi\widehat{X}}J_*\chi\widehat{U}.\end{aligned}$$

Since J is conformal Riemannian submersion, the by using equations (2.16) and from Lemma 2.1, we finally have

$$\begin{aligned}&J_*(\nabla_{n\widehat{X}}\chi\widehat{U})\\ &=-(cos^2\theta_1+cos^2\theta_2)J_*\{n\mathscr{T}_{t\widehat{X}}\omega P_1\widehat{U}+\chi\mathscr{V}\nabla_{t\widehat{X}}\omega P_1\widehat{U}+n\mathscr{A}_{n\widehat{X}}\omega P_1\widehat{U}+\chi\mathscr{V}\nabla_{n\widehat{X}}\omega P_1\widehat{U}\}\\ &\quad+J_*\{n\mathscr{T}_{t\widehat{X}}\omega P_1\widehat{U}+\chi\mathscr{V}\nabla_{t\widehat{X}}\omega P_1\widehat{U}+n\mathscr{A}_{n\widehat{X}}\omega P_1\widehat{U}+\chi\mathscr{H}\nabla_{n\widehat{X}}\omega P_1\widehat{U}\}\\ &\quad-J_*\{\chi\mathscr{T}_{t\widehat{X}}\chi\omega P_2\widehat{U}+n\mathscr{H}\nabla_{t\widehat{X}}\chi\omega P_2\widehat{U}+\chi\mathscr{T}_{t\widehat{X}}\chi\omega P_3\widehat{U}+n\mathscr{H}\nabla_{t\widehat{X}}\chi\omega P_3\widehat{U}\}\\ &\quad+n\widehat{X}(\ln\lambda)J_*\chi\omega P_2\widehat{U}+\chi\omega P_2\widehat{U}(\ln\lambda)J_*n\widehat{X}-g_1(n\widehat{X},\chi\omega P_2\widehat{U})J_*(grad\ln\lambda)\\ &\quad+n\widehat{X}(\ln\lambda)J_*\chi\omega P_3\widehat{U}+\chi\omega P_3\widehat{U}(\ln\lambda)J_*n\widehat{X}-g_1(n\widehat{X},\chi\omega P_3\widehat{U})J_*(grad\ln\lambda)\\ &\quad-J_*\{\chi\mathscr{A}_{n\widehat{X}}\chi\omega P_2\widehat{U}+\chi\mathscr{A}_{n\widehat{X}}\chi\omega P_3\widehat{U}-\mathscr{H}\nabla_{t\widehat{X}}\chi\widehat{U}\}\\ &\quad+J_*(\nabla_{\widehat{X}}\widehat{U})+\nabla^J_{\phi\widehat{X}}J_*\chi\widehat{U}-\nabla^J_{n\widehat{X}}J_*\chi\omega P_2\widehat{U}-\nabla^J_{n\widehat{X}}J_*\chi\omega P_3\widehat{U},\end{aligned}$$

which completes the proof of theorem.

6 Decomposition Theorems

The following conclusion from [29] is recalled in this section, and other decomposition theorem is discussed utilizing earlier proof. Let's say that on the manifold $M=\bar{B}_1\times\bar{B}_2$, g is a Riemannian metric. Then

(i) $M=\bar{B}_1\times_\lambda\bar{B}_2$ is a locally product if and only if $\bar{B}_1$ and $\bar{B}_2$ are totally geodesic foliations,

(ii) a warped product $\bar{B}_1\times_\lambda\bar{B}_2$ if and only if $\bar{B}_1$ is a totally geodesic foliation and $\bar{B}_2$ is a spherics foliation, i.e., it is umbilic and its mean curvature vector field is parallel,

(ii) $M=\bar{B}_1\times_\lambda\bar{B}_2$ is a twisted product if and only if $\bar{B}_1$ is a totally geodesic foliation and $\bar{B}_2$ is a totally umbilic foliation.

The fact that $J:(\bar{B}_1,\phi,\xi,\eta,g_1)\to(\bar{B}_2,g_2)$ is $\mathscr{QBSCS}$ ensures the existence of three orthogonal complementary distributions $\mathfrak{D}$, $\mathfrak{D}_{\theta_1}$, and $\mathfrak{D}_{\theta_2}$, all of which meet the previously stated characteristics of being integrable and totally geodesic. The logical next step is to search for the circumstances in which the total space $\bar{B}_1$ transforms into locally twisted product manifolds. We now present the following outcome.

Theorem 6.1 *Let J be a $\mathscr{QBSCS}$ from Kenmotsu manifold (M,ϕ,ξ,η,g_1) onto a RM (M_2,g_2). Then $\bar{B}_1$ is locally twisted product of the form $\bar{B}_{1(kerJ_*)}\times\bar{B}_{1(kerJ_*)^\perp}$ if and only if*

$$\frac{1}{\lambda^2}g_2((\nabla f_*)(\widehat{U},\chi\widehat{V}),f_*n\widehat{X}) = g_1(\mathscr{T}_{\widehat{U}}\omega\widehat{V},n\widehat{X})+g_1(\widehat{\mathscr{V}}\nabla_{\widehat{U}}\omega\widehat{V}+\mathscr{T}_{\widehat{U}}\chi\widehat{V},t\widehat{X}) + \frac{1}{\lambda^2}g_2(\nabla^J_U\chi\widehat{V},J_*n\widehat{X}) \tag{6.44}$$

and

$$g(\widehat{X},\widehat{Y})H = -t\mathscr{A}_{\widehat{X}}t\widehat{Y}-\omega\nabla_{\widehat{X}}t\widehat{Y}-\omega\mathscr{A}_{\widehat{X}}n\widehat{Y}-\phi J_*(\nabla^J_{\widehat{X}}J_*n\widehat{Y})+\widehat{X}(\ln\lambda)tn\widehat{Y} + n\widehat{Y}(\ln\lambda)t\widehat{X}-t(G\ln\lambda)g(\widehat{X},n\widehat{Y})-\eta(\widehat{U})g_1(\widehat{X},\widehat{Y}), \tag{6.45}$$

where H is a mean curvature vector and for any $\widehat{U},\widehat{V}\in\Gamma(kerJ_*)$ and $\widehat{X},\widehat{Y}\in\Gamma(kerJ_*)^\perp$.

Proof. For any $\widehat{X}\in\Gamma(kerJ_*)^\perp$ and $\widehat{U},\widehat{V}\in\Gamma(kerJ_*)$ and using equations (2.3), (2.6), (2.4), (2.12) and (2.13), we have

$$g_1(\nabla_{\widehat{U}}\widehat{V},\widehat{X}) = g_1(\mathscr{T}_{\widehat{U}}\omega V,nX)+g_1(\mathscr{V}\nabla_{\widehat{U}}\omega V+\mathscr{T}_{\widehat{U}}\chi V,tX)-g_1(\mathscr{H}\nabla_{\widehat{U}}\chi V,nX)$$

From using formula (2.7), (2.16) and with conformality of RS J, the above equation finally takes the form

$$g_1(\nabla_{\widehat{U}}\widehat{V},\widehat{X}) = g_1(\mathscr{T}_{\widehat{U}}\omega\widehat{V},n\widehat{X})+g_1(\widehat{\mathscr{V}}\nabla_{\widehat{U}}\omega\widehat{V}+\mathscr{T}_{\widehat{U}}\chi\widehat{V},t\widehat{X}) - \frac{1}{\lambda^2}g_2((\nabla f_*)(\widehat{U},\chi\widehat{V}),f_*n\widehat{X})+\frac{1}{\lambda^2}g_2(\nabla^J_U\chi\widehat{V},J_*n\widehat{X})$$

It follows that the equation (6.44) satisfies if and only if $\bar{B}_{1(kerJ_*)}$ is totally geodesic. On the other hand, for $\widehat{U}\in\Gamma(kerJ_*)$ and $\widetilde{X},\widehat{Y}\in\Gamma(kerJ_*)^\perp$ with using equations (2.3), (2.4), (2.6) and (3.21), we get

$$g_1(\nabla_{\widehat{X}}\widehat{Y},\widehat{U}) = g_1(\nabla_{\widehat{X}}P\widehat{Y},\phi\widehat{U})+g_1(\mathscr{A}_{\widehat{X}}n\widehat{Y},\omega\widehat{U})+g_1(\mathscr{H}\nabla_{\widehat{X}}n\widehat{Y},\chi\widehat{\widehat{U}})-\eta(\widehat{U})g_1(\widehat{X},\widehat{Y}).$$

By using the equation (2.16) with definition of conformality of J, we deduce that

$$g_1(\nabla_{\widehat{X}}\widehat{Y},\widehat{U}) = -\frac{1}{\lambda^2}g_2((\nabla J_*)(\widehat{X},n\widehat{Y}),J_*\chi\widehat{U})+\frac{1}{\lambda^2}g_2(\nabla^J_{\widehat{X}}J_*n\widehat{Y},J_*\chi\widehat{U}) + g_1(\nabla_{\widehat{X}}P\widehat{Y},\phi\widehat{U})+g_1(\mathscr{A}_{\widehat{X}}n\widehat{Y},\omega\widehat{U})-\eta(\widehat{U})g_1(\widehat{X},\widehat{Y}).$$

Considering the (i) part of Lemma 2.1, above equation turns in to

$$\begin{aligned} g_1(\nabla_{\widehat{X}}\widehat{Y},\widehat{U}) = &\frac{1}{\lambda^2}g_2(\nabla^J_{\widehat{X}}J_*n\widehat{Y},J_*\chi\widehat{U}) + g_1(\nabla_{\widehat{X}}t\widehat{Y},\phi\widehat{U}) + g_1(\mathscr{A}_{\widehat{X}}n\widehat{Y},\omega\widehat{U}) \\ &- g_1(G\ln\lambda,\widehat{X})g_1(n\widehat{Y},\chi\widehat{U}) - g_1(G\ln\lambda,n\widehat{Y})g_1(\widehat{X},\chi\widehat{U}) \\ &+ g_1(G\ln\lambda,\chi\widehat{U})g_1(\widehat{X},n\widehat{Y}) - \eta(\widehat{U})g_1(\widehat{X},\widehat{Y}). \end{aligned}$$

By direct calculation, finally we get

$$\begin{aligned} g_1(\widehat{X},\widehat{Y})H = &-t\mathscr{A}_{\widehat{X}}t\widehat{Y} - \omega\nabla_{\widehat{X}}t\widehat{Y} - \omega\mathscr{A}_{\widehat{X}}n\widehat{Y} - \phi J_*(\nabla^J_{\widehat{X}}J_*n\widehat{Y}) + \widehat{X}(\ln\lambda)tn\widehat{Y} \\ &+ n\widehat{Y}(\ln\lambda)t\widehat{X} - t(G\ln\lambda)g_1(\widehat{X},n\widehat{Y}) + \eta(\widehat{U})g_1(\phi\widehat{X},\widehat{Y}). \end{aligned}$$

From the above equation we conclude that $\bar{B}_{1(kerJ_*)^\perp}$ is totally umbilical if and only if equation (6.45) satisfied.

Declaration and Statement:

Conflict of Interest:
The authors declare that there is no conflict of interest.

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References

[1] M. A. Akyol and Y. Gunduzalp., Hemi-slant submersions from almost product Riemannian manifolds, Gulf J. Math., 4(3) (2016), 15-27.

[2] M. A. Akyol., Conformal semi-slant submersions, International Journal of Geometric Methods in Modern Physics, 14(7) (2017), 1750114.

[3] M. A. Akyol and B. Sahin., Conformal slant submersions, Hacettepe Journal of Mathematics and Statistics, 48(1) (2019), 28-44.

[4] M. A. Akyol and B. Sahin., Conformal anti-invariant submersions from almost Hermitian manifolds, Turkish Journal of Mathematics, 40 (2016), 43-70.

[5] M. A. Akyol and B. Sahin., Conformal semi-invariant submersions, Communications in Contemporary Mathematics, 19 (2017), 1650011.

[6] P. Baird and J. C. Wood., Harmonic Morphisms Between Riemannian Manifolds, London Mathematical Society Monographs, 29, Oxford University Press, The Clarendon Press. Oxford, (2003).

[7] J.-P. Bourguignon and H. B. Lawson., Jr., Stability and isolation phenomena for YangMills fields, Comm. Math. Phys., 79 (1981), no. 2, 189–230. http://projecteuclid.org/euclid.cmp/1103908963. ¨

[8] D. Chinea., Almost contact metric submersions, Rend. Circ. Mat. Palermo, 34(1) (1985), 89-104.

[9] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez and M. Fernandez., Slant submanifolds in Sasakian manifolds, Glasg. Math. J., 42 (1) (2000), 125-138.

[10] I. K. Erken and C. Murathan., On slant Riemannian submersions for cosymplectic manifolds, Bull. Korean Math. Soc., 51(6) (2014), 1749-1771.

[11] M. Falcitelli, S. Ianus and A. M. Pastore., Riemannian submersions and Related Topics, World Scientific, River Edge, NJ, (2004).

[12] A. Gray., Pseudo-Riemannian almost product manifolds and submersions, J. Math. Mech., 16 (1967), 715–737.

[13] B. Fuglede., Harmonic morphisms between Riemannian manifolds, Annales de l'institut Fourier (Grenoble), 28 (1978), 107-144.

[14] S. Gudmundsson., The geometry of harmonic morphisms, Ph.D. thesis, University of Leeds, (1992).

[15] S. Gudmundsson and J. C. Wood, Harmonic morphisms between almost Hermitian manifolds, Boll. Un. Mat. Ital. B (7) 11 (1997), no. 2, suppl., 185–197.

[16] Y. Gunduzalp., Semi-slant submersions from almost product Riemannian manifolds, Demonstratio Mathematica, 49(3) (2016), 345-356.

[17] Y. Gunduzalp and M. A. Akyol., Conformal slant submersions from cosymplectic manifolds, Turkish Journal of Mathematics, 48 (2018), 2672-2689.

[18] S. Ianu¸s and M. Vi¸sinescu., Space-time compaction and Riemannian submersions, In: Rassias, G.(ed.) The Mathematical Heritage of C. F. Gauss, World Scientific, River Edge (1991), 358-371.

[19] S. Ianu¸s and M. Vi¸sinescu., Kaluza-Klein theory with scalar fields and generalised Hopf manifolds, Classical Quantum Gravity, 4 (1987), no. 5, 1317–1325. http://stacks.iop.org/0264-9381/4/ 1317.

[20] T. Ishihara., A mapping of Riemannian manifolds which preserves harmonic functions, Journal of Mathematics of Kyoto University, 19 (1979), 215-229.

[21] Kenmotsu, K. (1972). A class of almost contact Riemannian manifolds. Tohoku Math., 24, 93-103.

[22] M. T. Mustafa., Applications of harmonic morphisms to gravity, J. Math. Phys., 41 (2000), 6918-6929.

[23] Y. Ohnita., On pluriharmonicity of stable harmonic maps, J. London Math. Soc. (2) 2 (1987), 563-568. 2.2

[24] B. O'Neill., The fundamental equations of a submersion, Michigan Math. J., 13 (1966), 459–469. http://projecteuclid.org/euclid.mmj/1028999604.

[25] K. S. Park and R. Prasad., Semi-slant submersions, Bull. Korean Math. Soc. 50(3) (2013), 951-962.

[26] R. Prasad, S. S. Shukla and S. Kumar., On Quasi bi-slant submersions, Mediterr. J. Math., 16 (2019), 155. https://doi.org/10.1007/s00009-019-1434-7.

[27] R. Prasad, M. A. Akyol, P. K. Singh and S. Kumar., On Quasi bi-slant submersions from Kenmotsu manifolds onto any Riemannian manifolds, Journal of Mathematical Extension, 8(16) (2021).

[28] R. Prasad and S. Kumar., Conformal anti-invariant submersions from nearly Kaehler Manifolds, Palestine Journal of Mathematics, 8(2) (2019).

[29] R. Ponge and H. Reckziegel., Twisted products in pseudo-Riemannian geometry, Geom. Dedicata, (1993), 48(1):15-25.

[30] Tanno, S. (1069). The automorphism groups of almost contact metric manfolds. Tohoku Math., J. 21, 21-38.

[31] B. Sahin., Anti-invariant Riemannian submersions from almost Hermitian manifolds, Central European J. Math., 3 (2010), 437-447.

[32] B. Sahin., Semi-invariant Riemannian submersions from almost Hermitian manifolds, Canad. Math. Bull., 56 (2013), 173-183.

[33] B. Şahin, Slant submersions from almost Hermitian manifolds, Bull. Math. Soc. Sci. Math. Roumanie. 1 (2011) 93-105.

[34] B.Sahin, M. A. Akyol Conformal Anti-Invariant Submersion From Almost Hermitian Manifolds, Turk J Math (2016) 40: 43 – 70

[35] M. Shuaib and T. Fatima. A note on conformal hemi-slant submersions, Afr. Mat. 34, 4 (2023).

[36] Sumeet Kumar et al., Conformal hemi-slant submersions from almost hermitian manifolds, Commun. Korean Math. Soc., 35 (2020), No. 3, pp. 999–1018 https://doi.org/10.4134/CKMS.c190448 pISSN: 1225-1763 / eISSN: 2234-3024.

[37] H. M. Tastan, B. Şahin and Ş. Yanan, Hemi-slant submersions, Mediterr. J. Math. 13(4) (2016) 2171-2184.

[38] B. Watson., Almost Hermitian submersions, J. Differential Geometry, vol. 11, no. 1, pp. (1976), 147–165. http://projecteuclid.org/euclid.jdg/1214433303

[39] B. Watson., G, G'-Riemannian submersions and nonlinear gauge field equations of general relativity, In: Rassias, T. (ed.) Global Analysis - Analysis on manifolds, dedicated M. Morse. Teubner-Texte Math., 57 (1983), 324-349, Teubner, Leipzig.