Geometry of Conformal Quasi Bi-slant Submersions

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Abstract In this paper, we study conformal quasi bi-slant submersions from almost product Riemannian manifolds onto Riemannian manifolds as a generalization of bi-slant submersions and hemi-slant submersions. We discuss integrability conditions for distributions with the study of geometry of leaves of the distributions. Also, we discuss pluriharmonicity for conformal quasi bislant submersions.

Keywords: Almost Product Riemannian manifold, Riemannian submersions, bi-slant submersions, quasi bi-slant submersions.

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هندسة الغمرات شبه المائلة شبه المتماثلة

الملخص: في هذا البحث، قمنا بدراسة الغمر المطابق شبه ثنائي المائل من المشعبات الريمانية المنتجة تقريبا إلى المشعبات الريمانية كتعميم للغمرات ثنائية المائلة والغمرات نصف المائلة. نناقش شروط التكامل للتوزيعات مع دراسة هندسة أوراق التوزيعات. نناقش أيضًا تعدد التناغم في الغمرات شبه الثنائية المتطابقة.

1 Introduction

Both mathematics and physics employ immersions and submersions extensively. Yang-Mills theory ([6], [29]), Kaluza-Klein theory ([12], [15]) are the significant application of submersion. The characteristics of slant submersions have become a fascinating topic in differential geometry, as well as in complex and contact geometry. The concept of Riemannian submersion between Riemannian manifolds has been extensively studied by prominent mathematicians such as B. O'Neill [17] and A. Gray [8], who independently made significant contributions to this field. In 1976, B. Watson [28], introduced the notion of almost Hermitian submersions, which focuses on the submersion between almost Hermitian manifolds. Since then, they have been widely used in differential geometry to study Riemannian manifolds having differentiable structures [26].

D. Chinea [7] introduced the concept of almost contact Riemannian submersions between almost contact metric manifolds. In his work, Chinea extensively examined the differential geometric aspects of the fiber space, base space, and total space involved in these submersions. A step forward, R Prasad et. al. studied quasi bi-slant submersions from almost contact metric manifolds onto Riemannian manifolds [21], [19], [14]. As a generalization of Riemannian submersions, Fuglede [9] and Ishihara [13], separately studied horizontally conformal submersions. Later on, many authors investigated different kinds of conformal Riemannian submersions like conformal anti-invariant submersions ([4], conformal slant submersions [3], conformal semi-slant submersions ([2], [11], [18]) and conformal hemi-slant submersions ([27], [1]) etc. from almost Hermitian manifolds onto a Riemannian manifold. Most of these Riemannian submersions and conformal submersions are also studied from almost contact metric manifolds onto a Riemannian manifold.

In this paper, we study conformal quasi bi-slant submersions from locally product Riemannian manifold onto a Riemannian manifold. This paper is divided into six sections. Section 2 contains brief history of Riemannian and conformal submersions. Also, we recall almost product Riemannian manifolds and, in particular, locally product Riemannian manifolds. In section 3, we investigate some fundamental results for conformal quasi bi-slant submersions from locally product Riemannian manifolds onto a Riemannian manifold those are required for our main sections. The results of integrability and totally geodesicness of distributions are presented in Section 4. In section 5, we discuss some decomposition theorems and also conditions under which locally product Riemannian manifold turns into locally twisted product manifold. While last section is devoted to the study of pluriharmonicity of conformal quasi bi-slant submersion.

Note: We will use some abbreviations throughout the paper as follows: locally product Riemannian manifold- LPR manifold, Conformal quasi bi-slant submersion- CQBS submersion,

2 Preliminaries

In this section, we will discuss the concept of almost product Riemannian manifold and also, Riemannian submersions and conformal submersions between two Riemannian manifolds with some basic facts and results. These concepts have been previously introduced in the earlier work in this field, so we mentioned them in quotation and proper references have been provided to acknowledge their contributions. Furthermore, the definitions have been restated here to ensure clarity and facilitate a comprehensive understanding of the concepts presented in this study.

"An n-dimensional manifold $\bar{Q}$ with (1,1) type tensor field F such that

$$F^2 = I, (F \neq I), \tag{2.1}$$

is called an almost product manifold with almost structure F. There exists a Riemannian metric g on an almost product manifold which is compatible with the structure F in the sense that

$$g(FU, FV) = g(U, V),$$

(2.2)

for any $U, V \in \Gamma(TM)$, then $(\bar{Q}, g, F)$ is called an almost product Riemannian manifold. The covariant derivative of F defined by

$$(\nabla_U FV) = \nabla_U FV - F\nabla_U V$$

(2.3)

for any vector fields $U, V \in \Gamma(\bar{T}Q)$. The manifold is called locally product Riemannian (LPR) manifold if F is parallel with respect to connection ∇.e.,

$$(\nabla\_U\ F)V = 0 \quad (2.4)$$

for any vector fields $U, V \in \Gamma(\bar{T}Q)$.

Let $\Psi: (\bar{Q}_1, g_1, F) \to (\bar{Q}_2, g_2)$ be a Riemannian submersion. A vector field $\bar{X}$ on $\bar{Q}_1$ is called a basic vector field if $\bar{X} \in \Gamma(\ker \Psi_*)^{\perp}$ and Ψ-related with a vector field $\bar{X}$ on $\bar{Q}_2$ i.e., $\Psi_*(\bar{X}(q)) = \bar{X}\Psi(q)$ for $q \in \bar{Q}_1$."

The formulae given by O'Neill [17] of two (1,2) tensor fields $\mathcal{T}$ and $\mathcal{A}$, plays a crucial role in the theory of submersions

(2.5)

$$\mathcal{A}_{E_1}F_1 = \mathcal{H}\nabla_{\mathcal{H}E_1}\mathcal{V}F_1 + \mathcal{V}\nabla_{\mathcal{H}E_1}\mathcal{H}F_1, \quad (2.6)$$
$$\mathcal{T}_{E_1}F_1 = \mathcal{H}\nabla_{\mathcal{V}E_1}\mathcal{V}F_1 + \mathcal{V}\nabla_{\mathcal{V}E_1}\mathcal{H}F_1,$$

for any $E_1, F_1 \in \Gamma(T\bar{Q}_1)$ and ∇ is Levi-Civita connection of g_1. Note that a Riemannian submersion $\Psi: (\bar{Q}_1, g_1, F) \to (\bar{Q}_2, g_2)$ has totally geodesic fibers if and only if $\mathcal{T}$ vanishes identically. From (2.5) and (2.6), we can deduce (2.7)

(2.8)

$$\nabla_{\tilde{U}_1}\tilde{V}_1 = \mathcal{T}_{\tilde{U}_1}\tilde{V}_1 + \mathcal{V}\nabla_{\tilde{U}_1}\tilde{V}_1 \quad (2.9)$$
$$\nabla_{\tilde{U}_1}\tilde{X}_1 = \mathcal{T}_{\tilde{U}_1}\tilde{X}_1 + \mathcal{H}\nabla_{\tilde{U}_1}\tilde{X}_1 \quad (2.10$$
$$\nabla_{\tilde{X}_1}\tilde{U}_1 = \mathcal{A}_{\tilde{X}_1}\tilde{U}_1 + \mathcal{V}_1\nabla_{\tilde{X}_1}\tilde{U}_1$$
$$\nabla_{\tilde{X}_1}\tilde{Y}_1 = \mathcal{H}\nabla_{\tilde{X}_1}\tilde{Y}_1 + \mathcal{A}_{\tilde{X}_1}\tilde{Y}_1$$

for any vector fields $\tilde{U}_1, \tilde{V}_1 \in \Gamma(\ker \Psi_*)$ and $\tilde{X}_1, \tilde{Y}_1 \in \Gamma(\ker \Psi_*)^{\perp}$.
It is observe that $\mathcal{T}$ and $\mathcal{A}$ are skew-symmetric, that is,

$$g(\mathcal{A}_{\tilde{X}} E_1, F_1) = -g(E_1, \mathcal{A}_{\tilde{X}} F_1), g(\mathcal{T}_{\tilde{V}} E_1, F_1) = -g(E_1, \mathcal{T}_{\tilde{V}} F_1), \quad (2.11)$$

for any vector fields $E_1, F_1 \in \Gamma(T_p \bar{Q}_1)$. It is also observed that the restriction of $\mathcal{T}$ to the vertical distribution $\mathcal{T}|_{V\times V}$ is exactly the second fundamental form of the fibres of Ψ. Since $\mathcal{T}_V$ is skew-symmetric we say Ψ has totally geodesic fibres if and only if $\mathcal{T} = 0$. For the special case when Ψ is horizontally conformal submersion we have "

Proposition 2.1. [10] Let $\Psi: (\bar{Q}_1, g_1) \to (\bar{Q}_2, g_2)$ be a horizontally conformal submersion with dilation λ and X, Y be the horizontal vectors, then

$$\mathcal{A}_X Y = \frac{1}{2}\left[\mathcal{V}[X, Y] - \lambda^2 g_1(X, Y) \mathrm{grad}\left(\frac{1}{\lambda^2}\right)\right]$$

(2.12)

We see that the skew-symmetric part of measures the obstruction integrability of the horizontal distribution $(\ker \Psi_*)^{\perp}$.

Definition 2.1. A horizontally conformally submersion $\Psi: \bar{Q}_1 \to \bar{Q}_2$ is called horizontally homothetic if the gradient of its dilation λ is vertical, i.e.,

(2.13)

$$\mathcal{H}(\mathrm{grad}\, \lambda) = 0,$$

where $\mathcal{H}$ is the orthogonal complementary distribution to $\nu = \ker \Psi_*$ in $\Gamma(T_p \bar{Q}_1)$.
The second fundamental form of smooth map Ψ is given by

(2.14)

$$(\nabla \Psi_*)(\tilde{U}_1, \tilde{V}_1) = \nabla^{\Psi}_{\tilde{U}_1} \Psi_* \tilde{V}_1 - \Psi_* \nabla_{\tilde{U}_1} \tilde{V}_1,$$

and the map be totally geodesic if $(\nabla \Psi_*)(\tilde{U}_1, \tilde{V}_1) = 0$ for all $\tilde{U}_1, \tilde{V}_1 \in \Gamma(T_p \bar{Q}_1)$, where ∇ and $\nabla \Psi_*$ are Levi-Civita and pullback connections."

Now, we recall the following lemma for our main section.

Lemma 2.1. Let $\Psi: \bar{Q}_1 \to \bar{Q}_2$ be a horizontal conformal submersion. Then, we have

(i) $(\nabla\Psi_*)(\tilde{X}_1, \tilde{Y}_1) = \tilde{X}_1(\ln\lambda)\Psi_*(\tilde{Y}_1) + \tilde{Y}_1(\ln\lambda)\Psi_*(\tilde{X}_1) - g_1(\tilde{X}_1, \tilde{Y}_1)\Psi_*(\mathrm{grad}\ln\lambda)$,

(ii) $(\nabla\Psi_*)(\tilde{U}_1, \tilde{V}_1) = -\Psi_*(\mathcal{T}_{\tilde{U}_1}\tilde{V}_1)$,

(iii) $(\nabla\Psi_*)(\tilde{X}_1, \tilde{U}_1) = -\Psi_*(\nabla_{\tilde{X}_1}\tilde{U}_1) = -\Psi_*(\mathcal{A}_{\tilde{X}_1}\tilde{U}_1)$,

for any horizontal vector fields $\tilde{X}_1, \tilde{Y}_1$ and vertical vector fields $\tilde{U}_1, \tilde{V}_1$[5].

3 Conformal quasi bi-slant submersions

First of all we are giving in this sections some definitions that will useful througout the text.

Definition 3.1. [28] Let $\Psi: (\bar{Q}_1, g_{Q_1}) \to (\bar{Q}_2, g_{Q_2})$ be a smooth map between two Riemannian manifolds having dimensions m_1 and m_2, respectively. Then Ψ is called horizontally weakly conformal or semi conformal at $x \in \bar{Q}_1$ if either

(i) $\Psi_{*x} = 0$, or

(ii) Ψ_{*x} maps horizontal space $\mathcal{H}_x = (\ker(\Psi_{*x}))^{\perp}$ conformally onto $T_{\Psi*}(N)$ i (3.1)
is surjective and there exits a number $\Lambda(x) \neq 0$ such that

$$g_N(\Psi_{*x}X, \Psi_{*x}Y) = \Lambda(x)g(X, Y),$$

for any $X, Y \in \mathcal{H}_x$.

Equation (3.1) can be re-written as

$$(\Psi_* g_N)_x \mid_{\mathcal{H}_x\times\mathcal{H}_x} = \Lambda(x)g(x) \mid_{\mathcal{H}_x\times\mathcal{H}_x}.$$

A point x satisfies (i) in above definition if and only if it is a critical point of Ψ. A point, satisfying (ii) is called a regular point. At a critical point, Ψ_{*x} has rank 0 ; at a regular point, Ψ_{*x} has rank n and Ψ defines a submersion. The number $\lambda(x)$ is called the square dilation (of Ψ at x); it is necessarily nonnegative. Its square root $\lambda(x) = \sqrt{\Lambda(x)}$ is called the dilation of Ψ at x. The map Ψ is called horizontally weakly conformal or semi conformal on $\bar{Q}_1$ if it is horizontally weakly conformal at every point of M. It is clear that if Ψ has no critical points, then we call it a (horizontally) conformal submersion.

Definition 3.2 . Let us suppose that $(\bar{Q}_1, g_1, F)$ be an almost product manifold and $(\bar{Q}_2, g_2)$ be a Riemannian manifold. A Riemannian submersion Ψ from $\bar{Q}_1$ onto $\bar{Q}_2$ is called a conformal quasi bi-slant $(CQBS)$ submersion if there exists three mutually orthogonal distributions $\mathfrak{D}^{\mathfrak{T}}, \mathfrak{D}^{\theta_1}$ and $\mathfrak{D}^{\theta_2}$ such that

(i) $\ker \Psi_* = \mathfrak{D}^{\mathfrak{T}} \oplus \mathfrak{D}^{\theta_1} \oplus \mathfrak{D}^{\theta_2}$,

(ii) $\mathfrak{D}^{\mathfrak{T}}$ is invariant. i.e., $F\mathfrak{D}^{\mathfrak{T}} = \mathfrak{D}^{\mathfrak{T}}$,

(iii) $F\mathfrak{D}^{\theta_1} \perp \mathfrak{D}^{\theta_2}$ and $F\mathfrak{D}^{\theta_2} \perp \mathfrak{D}^{\theta_1}$,

(iv) for any non-zero vector field $\tilde{V}_1 \in \left(\mathfrak{D}^{\theta_1}\right)_p, p \in \bar{Q}_1$ the angle θ_1 between $\left(\mathfrak{D}^{\theta_1}\right)_p$ and $F\tilde{V}_1$ is constant and independent of the choice of the point p and $\tilde{V}_1 \in \left(\mathfrak{D}^{\theta_1}\right)_p$,

(v) for any non-zero vector field $\tilde{V}_1 \in \left(\mathfrak{D}^{\theta_2}\right)_q, q \in \bar{Q}_1$ the angle θ_2 between $\left(\mathfrak{D}^{\theta_2}\right)_q$ and $F\tilde{V}_1$ is constant and independent of the choice of the point q and $\tilde{V}_1 \in \left(\mathfrak{D}^{\theta_2}\right)_q$.

If we denote the dimensions of $\mathfrak{D}^{\mathfrak{T}}, \mathfrak{D}^{\theta_1}$ and $\mathfrak{D}^{\theta_2}$ by m_1, m_2 and m_3 respectively, then we have the following observations:

(i) If $m_1 \neq 0, m_2 = 0$ and $m_3 = 0$, then Ψ is an invariant submersion.

(ii) If $m_1 \neq 0, m_2 \neq 0, 0 < \theta_1 < \frac{\pi}{2}$ and $m_3 = 0$, then Ψ is a proper semi-slant submersion.

(iii) If $m_1 = 0, m_2 = 0$ and $m_3 \neq 0, 0 < \theta_2 < \frac{\pi}{2}$, then Ψ is a slant submersion with slant angle θ_2.

(iv) If $m_1 = 0, m_2 \neq 0, 0 < \theta_1 < \frac{\pi}{2}$ and $m_3 \neq 0, \theta_2 = \frac{\pi}{2}$, then Ψ proper hemi-slant submersion.

(v) If $m_1 = 0, m_2 \neq 0, 0 < \theta_1 < \frac{\pi}{2}$ and $m_3 \neq 0, 0 < \theta_2 < \frac{\pi}{2}$, then Ψ is proper bi-slant submersion with slant angles θ_1 and θ_2.

(vi) If $m_1 \neq 0, m_2 \neq 0, 0 < \theta_1 < \frac{\pi}{2}$ and $m_3 \neq 0, 0 < \theta_2 < \frac{\pi}{2}$, then Ψ is proper quasi bi-slant submersion with slant angles θ_1 and θ_2.

Hence, it is clear that CQBS submersions are generalized version of conformal quasi hemi-slant submersions.

Let Ψ be a CQBS submersion from an almost product Riemannian manifold $(\bar{Q}_1, g_1, F)$ onto a Riemannian manifold $(\bar{Q}_2, g_2)$. Then, for any $U \in (\ker \Psi_*)$, we have

$$\widetilde{U} = \mathfrak{A}\widetilde{U} + \mathfrak{B}\widetilde{U} + \mathfrak{C}\widetilde{U}, \tag{3.2}$$

where $\mathfrak{A}, \mathfrak{B}$ and $\mathfrak{C}$ are the projections morphism onto $\mathfrak{D}^{\mathfrak{T}}, \mathfrak{D}^{\theta_1}$, and $\mathfrak{D}^{\theta_2}$, respectively. Now, for any $\widetilde{U} \in (\ker \Psi_*)$, we have

$$F\widetilde{U} = \xi\widetilde{U} + \eta\widetilde{U} \tag{3.3}$$

where $\xi\widetilde{U} \in \Gamma(\ker \Psi_*)$ and $\eta\widetilde{U} \in \Gamma(\ker \Psi_*)^{\perp}$. From equations (3.2) and (3.3), we have

$$\begin{aligned} F\widetilde{U} &= F(\mathfrak{A}\widetilde{U}) + F(\mathfrak{B}\widetilde{U}) + F(\mathfrak{C}\widetilde{U}) \\ &= \xi(\mathfrak{A}\widetilde{U}) + \eta(\mathfrak{A}\widetilde{U}) + \xi(\mathfrak{B}\widetilde{U}) + \eta(\mathfrak{B}\widetilde{U}) + \xi(\mathfrak{C}\widetilde{U}) + \eta(\mathfrak{C}\widetilde{U}). \end{aligned}$$

Since $F\mathfrak{D}^{\mathfrak{T}} = \mathfrak{D}^{\mathfrak{T}}$ and $\eta(\mathfrak{A}\widetilde{U}) = 0$, we have

$$F\widetilde{U} = \xi(\mathfrak{A}\widetilde{U}) + \xi(\mathfrak{B}\widetilde{U}) + \eta(\mathfrak{B}\widetilde{U}) + \xi(\mathfrak{C}\widetilde{U}) + \eta(\mathfrak{C}\widetilde{U}).$$

Hence, we have the decomposition as

$$F(\ker \Psi_*) = \xi\mathfrak{D}^{\mathfrak{T}} \oplus \xi\mathfrak{D}^{\theta_1} \oplus \xi\mathfrak{D}^{\theta_2} \oplus \eta\mathfrak{D}^{\theta_1} \oplus \eta\mathfrak{D}^{\theta_2}.$$

(3.4)

From equations (3.4), we get

$$(\ker \Psi_*)^\perp = \eta\mathfrak{D}^{\theta_1} \oplus \eta\mathfrak{D}^{\theta_2} \oplus \mu, \tag{3.5}$$

where μ is the orthogonal complement to $\eta\mathfrak{D}^{\theta_1} \oplus \eta\mathfrak{D}^{\theta_2}$ in $(\ker \Psi_*)^\perp$ and μ is invariant with respect to F. Now, for any $\tilde{X} \in \Gamma(\ker \Psi_*)^\perp$, we have

$$F\tilde{X} = P\tilde{X} + L\tilde{X} \tag{3.6}$$

where $P\tilde{X} \in \Gamma(\ker \Psi_*)$ and $L\tilde{X} \in \Gamma(\mu)$.

Lemma 3.1. Let $(\bar{Q}_1, g_1, F)$ be an almost product Riemannian manifold and $(\bar{Q}_2, g_2)$ be a Riemannian manifold. If $\Psi: \bar{Q}_1 \to \bar{Q}_2$ is a CQBS submersion, then we have

$$\widetilde{U} = \xi^2\widetilde{U} + P\xi\widetilde{U}, \eta\xi\widetilde{U} + L\eta\widetilde{U} = 0,$$
$$\tilde{X} = \eta P\tilde{X} + L^2\tilde{X}, \xi P\tilde{X} + PL\tilde{X} = 0,$$

for $\widetilde{U} \in \Gamma(\ker \Psi_*)$ and $\tilde{X} \in \Gamma(\ker \Psi_*)^\perp$.

Proof. By using equations (2.1), (3.3) and (3.6), we get the desired results.
Since $\Psi: \bar{Q}_1 \to \bar{Q}_2$ is a CQBS submersion, Here we give some useful results that will be used throughout the paper.

Lemma 3.2. [19] Let Ψ be a CQBS submersion from an almost product Riemannian manifold $(\bar{Q}_1, g_1, F)$ onto a Riemannian manifold $(\bar{Q}_2, g_2)$, then we have

(i) $\xi^2\widetilde{U} = \cos^2\theta_1\widetilde{U}$,

(ii) $g_1(\xi\widetilde{U}, \xi\tilde{V}) = \cos^2\theta_1 g_1(\widetilde{U}, \tilde{V})$,

(iii) $g(\eta\widetilde{U}, \eta\tilde{V}) = \sin^2\theta_1 g_1(\widetilde{U}, \tilde{V})$,

for any vector fields $\widetilde{U}, \tilde{V} \in \Gamma(\mathfrak{D}^{\theta_1})$.

Lemma 3.3. [19] Let Ψ be a CQBS submersion from an almost product Riemannian manifold $(\bar{Q}_1, g_1, F)$ onto a Riemannian manifold $(\bar{Q}_2, g_2)$, then we have

(i) $\xi^2 \tilde{Z} = \cos^2 \theta_2 \tilde{Z}$,

(ii) $g_1(\xi\tilde{Z}, \xi\widetilde{W}) = \cos^2 \theta_2 g_1(\tilde{Z}, \widetilde{W})$,

(iii) $g_1(\eta\tilde{Z}, \eta\widetilde{W}) = \sin^2 \theta_2 g_1(\tilde{Z}, \widetilde{W})$,

for any vector fields $\tilde{Z}, \widetilde{W} \in \Gamma(\mathfrak{D}^{\theta_2})$.

The proof of above Lemmas is similar to the proof of the Theorem 3.5 of [19]. Thus, we omit the proofs.

Let $(\bar{Q}_2, g_2)$ be a Riemannian manifold and that $(\bar{Q}_1, g_1, F)$ is a LPR manifold. We now observe how the tensor fields $\mathcal{T}$ and $\mathcal{A}$ of a CQBS submersion $\Psi: (\bar{Q}_1, g_1, F) \to (\bar{Q}_2, g_2)$ are affected by the LPR structure on $\bar{Q}_1$.

Lemma 3.4. Let Ψ be a CQBS submersion from an almost product Riemannian manifold $(\bar{Q}_1, g_1, F)$ onto a Riemannian manifold $(\bar{Q}_2, g_2)$, then we have

$$\mathcal{A}_{\tilde{X}} P\tilde{Y} + \mathcal{H}\nabla_{\tilde{X}} L\tilde{Y} = \xi\mathcal{H}\nabla_{\tilde{X}}\tilde{Y} + P\mathcal{A}_{\tilde{X}}\tilde{Y} \tag{3.7}$$

$$\mathcal{V}\nabla_{\tilde{X}} P\tilde{Y} + \mathcal{A}_{\tilde{X}} L\tilde{Y} = \eta\mathcal{H}\nabla_{\tilde{X}}\tilde{Y} + L\mathcal{A}_{\tilde{X}}\tilde{Y}. \tag{3.8}$$

$$\mathcal{V}\nabla_{\tilde{X}}\xi\tilde{V} + \mathcal{A}_{\tilde{X}}\eta\tilde{V} = P\mathcal{A}_{\tilde{X}}\tilde{V} + \xi\mathcal{V}\nabla_{\tilde{X}}\tilde{V} \tag{3.9}$$

$$\mathcal{A}_{\tilde{X}}\xi\tilde{V} + \mathcal{H}\nabla_{\tilde{X}}\eta\tilde{V} = \mathcal{C}\mathcal{A}_{\tilde{X}}\tilde{V} + \eta\mathcal{V}\nabla_{\tilde{X}}\tilde{V}. \tag{3.10}$$

$$\mathcal{V}\nabla_{\tilde{V}} P\tilde{X} + \mathcal{T}_{\tilde{V}} L\tilde{X} = \xi\mathcal{T}_{\tilde{V}} L\tilde{X} + P\mathcal{H}\nabla_{\tilde{V}}\tilde{X} \tag{3.11}$$

$$\mathcal{T}_{\tilde{V}} P\tilde{X} + \mathcal{H}\nabla_{\tilde{V}} L\tilde{X} = \eta\mathcal{T}_{\tilde{V}}\tilde{X} + L\mathcal{H}\nabla_{\tilde{V}}\tilde{X}. \tag{3.12}$$

$$\mathcal{V}\nabla_{\tilde{U}}\xi\tilde{V} + \mathcal{T}_{\tilde{U}}\eta\tilde{V} = \mathfrak{B}\mathcal{T}_{\tilde{U}}\tilde{V} + \xi\mathcal{V}\nabla_{\tilde{U}}\tilde{V} \tag{3.13}$$

$$\mathcal{T}_{\tilde{U}}\xi\tilde{V} + \mathcal{H}\nabla_{\tilde{U}}\eta\tilde{V} = L\mathcal{T}_{\tilde{U}}\tilde{V} + \eta\mathcal{V}\nabla_{\tilde{U}}\tilde{V}, \tag{3.14}$$

for any vector fields $\tilde{U}, \tilde{V} \in \Gamma(\ker \Psi_*)$ and $\tilde{X}, \tilde{Y} \in \Gamma(\ker \Psi_*)^\perp$.

Proof. By the direct calculation, using (3.6), (2.10) and (2.3), we can easily obtain relations given by (3.7) and (3.8). Remaining relations can be obtained similarly by using (3.3), (3.6), (2.7)-(2.10) and (2.3).

Now, we discuss some basic results which are useful to explore the geometry of CQBS submersion $\Psi: \bar{Q}_1 \to \bar{Q}_2$. For this, define the following :

$$
\begin{aligned}
(\nabla_{\tilde{U}}\xi)\tilde{V} &= \mathcal{V}\nabla_{\tilde{U}}\xi\tilde{V} - \xi\mathcal{V}\nabla_{\tilde{U}}\tilde{V} && (3.15)\\
(\nabla_{\tilde{U}}\eta)\tilde{V} &= \mathcal{H}\nabla_{\tilde{U}}\eta\tilde{V} - \eta\mathcal{V}\nabla_{\tilde{U}}\tilde{V} && (3.16)\\
(\nabla_{\tilde{X}}P)\tilde{Y} &= \mathcal{V}\nabla_{\tilde{X}}P\tilde{Y} - P\mathcal{H}\nabla_{\tilde{X}}\tilde{Y} && (3.17)\\
(\nabla_{\tilde{X}}L)\tilde{Y} &= \mathcal{H}\nabla_{\tilde{X}}L\tilde{Y} - L\mathcal{H}\nabla_{\tilde{X}}\tilde{Y} && (3.18)
\end{aligned}
$$

for any vector fields $\tilde{U}, \tilde{V} \in \Gamma(\ker \Psi_*)$ and $\tilde{X}, \tilde{Y} \in \Gamma(\ker \Psi_*)^{\perp}$.

Lemma 3.5. Let $(\bar{Q}_1, g_1, F)$ be LPR manifold and $(\bar{Q}_2, g_2)$ be a Riemannian manifold. If $\Psi: \bar{Q}_1 \to \bar{Q}_2$ is a CQBS submersion, then we have

$$
\begin{aligned}
(\nabla_{\tilde{U}}\xi)\tilde{V} &= P\mathcal{T}_{\tilde{U}}\tilde{V} - \mathcal{T}_{\tilde{U}}\eta\tilde{V}\\
(\nabla_{\tilde{U}}\eta)\tilde{V} &= L\mathcal{T}_{\tilde{U}}\tilde{V} - \mathcal{T}_{\tilde{U}}\xi\tilde{V}\\
(\nabla_{\tilde{X}}P)\tilde{Y} &= \xi\mathcal{A}_{\tilde{X}}\tilde{Y} - \mathcal{A}_{\tilde{X}}L\tilde{Y}\\
(\nabla_{\tilde{X}}L)\tilde{Y} &= \eta\mathcal{A}_{\tilde{X}}\tilde{Y} - \mathcal{A}_{\tilde{X}}P\tilde{Y},
\end{aligned}
$$

for all vector fields $\tilde{U}, \tilde{V} \in \Gamma(\ker \Psi_*)$ and $\tilde{X}, \tilde{Y} \in \Gamma(\ker \Psi_*{}^{\perp})$.

Proof. By using equations (2.3), (2.7)- (2.10) and equations (3.15)-(3.18), we get the proof of the lemma.

If the tenors ξ and η are parallel with respect to the connection ∇ of $\bar{Q}_1$ then, we have

$$P\mathcal{T}_{\tilde{U}}\tilde{V} = \mathcal{T}_{\tilde{U}}\eta\tilde{V}, L\mathcal{T}_{\tilde{U}}\tilde{V} = \mathcal{T}_{\tilde{U}}\xi\tilde{V}$$

for any vector fields $\tilde{U}, \tilde{V} \in \Gamma(T\bar{Q}_1)$.

4 Integrability and totally geodesicness of distributions

Firslty, we are giving definition of Riemannian manifold.

A Riemannian metric on a smooth manifold M is a symmetric positive definite smooth 2-covariant tensor field g. A smooth manifold M equipped with a Riemannian metric g is called a Riemannian manifold and denoted by (M, g).

Since, $\Psi: \bar{Q}_1 \to \bar{Q}_2$ is a CQBS submersion, where $(\bar{Q}_1, g_1, F)$ representing a LPR manifold and $(\bar{Q}_2, g_2)$ denoting a Riemannian manifold. The existence of three mutually orthogonal distributions, including an invariant distribution $\mathfrak{D}$, a pair of slant distributions $\mathfrak{D}^{\theta_1}$ and $\mathfrak{D}^{\theta_2}$, is guaranteed by the definition of CQBS-submersion. We begin the subject of distributions integrability by determining the integrability of the slant distributions as follows:

Theorem 4.1. Let Ψ be a CQBS submersion from LPR manifold $(\bar{Q}_1, g_1, F)$ onto a Riemannian manifold $(\bar{Q}_2, g_2)$. Then slant distribution $\mathfrak{D}^{\theta_1}$ is integrable if and only if

$$\begin{aligned}
&\frac{1}{\lambda^2}\left\{g_2\left(\nabla^{\Psi}_{\widetilde{U}_1}\Psi_*\eta\tilde{V}_1 + \nabla^{\Psi}_{\tilde{V}_1}\Psi_*\eta\widetilde{U}_1, \Psi_*\eta\mathfrak{C}\tilde{Z}\right)\right\} \\
&= \frac{1}{\lambda^2}\left\{g_2\left((\nabla\Psi_*)\left(\widetilde{U}_1, \eta\tilde{V}_1\right) + (\nabla\Psi_*)\left(\tilde{V}_1, \eta\widetilde{U}_1\right), \Psi_*\eta\mathfrak{C}\tilde{Z}\right)\right\} \\
&- g_1\left(\nabla_{\widetilde{U}_1}\eta\xi\tilde{V}_1 - \nabla_{\tilde{V}_1}\eta\xi\widetilde{U}_1, \tilde{Z}\right) - g_1\left(\mathcal{T}_{\widetilde{U}_1}\eta\tilde{V}_1 - \mathcal{T}_{\tilde{V}_1}\eta\widetilde{U}_1, F\mathfrak{U}\tilde{Z} + \xi\mathfrak{C}\tilde{Z}\right),
\end{aligned} \tag{4.1}$$

for any $\widetilde{U}_1, \tilde{V}_1 \in \Gamma\left(\mathfrak{D}^{\theta_1}\right)$ and $\tilde{Z} \in \Gamma\left(\mathfrak{D}^{\mathfrak{T}} \oplus \mathfrak{D}^{\theta_2}\right)$.

Proof. For $\widetilde{U}_1, \tilde{V}_1 \in \Gamma\left(\mathfrak{D}^{\theta_1}\right)$ and $\tilde{Z} \in \Gamma\left(\mathfrak{D}^{\mathfrak{T}} \oplus \mathfrak{D}^{\theta_2}\right)$ with using equations (2.2), (2.3), (2.13) and (3.3), we get

$$\begin{aligned}
g_1\left(\left[\widetilde{U}_1, \tilde{V}_1\right], \tilde{Z}\right) = \quad & g_1\left(\nabla_{\widetilde{U}_1}\xi\tilde{V}_1, F\tilde{Z}\right) + g_1\left(\nabla_{\widetilde{U}_1}\eta\tilde{V}_1, F\tilde{Z}\right) \\
& - g_1\left(\nabla_{\tilde{V}_1}\xi\widetilde{U}_1, F\tilde{Z}\right) - g_1\left(\nabla_{\tilde{V}_1}\eta\widetilde{U}_1, F\tilde{Z}\right).
\end{aligned}$$

By using equations (2.3), (2.13) and (3.3), we have

$$\begin{aligned} g_1([\tilde{U}_1,\tilde{V}_1],\tilde{Z}) = \; & g_1(\nabla_{\tilde{U}_1}\xi^2\tilde{V}_1,\tilde{Z}) + g_1(\nabla_{\tilde{U}_1}\eta\xi\tilde{V}_1,\tilde{Z}) - g_1(\nabla_{\tilde{V}_1}\xi^2\tilde{U}_1,\tilde{Z}) \\ & -g_1(\nabla_{\tilde{V}_1}\eta\xi\tilde{U}_1,\tilde{Z}) + g_1(\nabla_{\tilde{U}_1}\eta\tilde{V}_1, F\mathfrak{A}\tilde{Z} + \xi\mathfrak{C}\tilde{Z} + \eta\mathfrak{C}\tilde{Z}) \\ & -g_1(\nabla_{\tilde{V}_1}\eta\tilde{U}_1, F\mathfrak{A}\tilde{Z} + \xi\mathfrak{C}\tilde{Z} + \eta\mathfrak{C}\tilde{Z}). \end{aligned}$$

Taking account the fact of Lemma 3.2 with using equation (2.8), we get

$$\begin{aligned} g_1([\tilde{U}_1,\tilde{V}_1],\tilde{Z}) = \; & \cos^2\theta_1 g_1([\tilde{U}_1,\tilde{V}_1],\tilde{Z}) + g_1(\nabla_{\tilde{U}_1}\eta\xi\tilde{V}_1 - \nabla_{\tilde{V}_1}\eta\xi\tilde{U}_1,\tilde{Z}) \\ & +g_1(\mathcal{T}_{\tilde{U}_1}\eta\tilde{V}_1 - \mathcal{T}_{\tilde{V}_1}\eta\tilde{U}_1, F\mathfrak{A}\tilde{Z} + \xi\mathfrak{C}\tilde{Z}) \\ & +g_1(\mathcal{H}\nabla_{\tilde{U}_1}\eta\tilde{V}_1 - \mathcal{H}\nabla_{\tilde{V}_1}\eta\tilde{U}_1, \eta\mathfrak{C}\tilde{Z}). \end{aligned}$$

By using formula (2.14) with Lemma 2.1, we finally get

$$\begin{aligned} & \sin^2\theta_1 g_1([\tilde{U}_1,\tilde{V}_1],\tilde{Z}) \\ & = \frac{1}{\lambda^2}\{g_2(\nabla^{\Psi}_{\tilde{U}_1}\Psi_*\eta\tilde{V}_1 - \nabla^{\Psi}_{\tilde{V}_1}\Psi_*\eta\tilde{U}_1, \Psi_*\eta\mathfrak{C}\tilde{Z})\} \\ & \;+ \frac{1}{\lambda^2}\left\{g_2\left((\nabla\Psi_*)(\tilde{U}_1,\eta\tilde{V}_1),\Psi_*\eta\mathfrak{C}\tilde{Z}\right) + g_2\left((\nabla\Psi_*)(\tilde{V}_1,\eta\tilde{U}_1),\Psi_*\eta\mathfrak{C}\tilde{Z}\right)\right\} \\ & \;+ g_1(\mathcal{T}_{\tilde{U}_1}\eta\tilde{V}_1 - \mathcal{T}_{\tilde{V}_1}\eta\tilde{U}_1, F\mathfrak{A}\tilde{Z} + \xi\mathfrak{C}\tilde{Z}) + g_1(\nabla_{\tilde{U}_1}\eta\xi\tilde{V}_1 - \nabla_{\tilde{V}_1}\eta\xi\tilde{U}_1,\tilde{Z}). \end{aligned}$$

In a similar way, we can examine the condition of integrability for slant distribution as follows:

Theorem 4.2. Let $\Psi: (\bar{Q}_1, g_1, F) \to (\bar{Q}_2, g_2)$ be a CQBS submersion, where $(\bar{Q}_1, g_1, F)$ a LPR manifold and $(\bar{Q}_2, g_2)$ a Riemannian manifold. Then slant distribution $\mathfrak{D}^{\theta_2}$ is integrable if and only if

$$\begin{aligned} - \; & \frac{1}{\lambda^2}\left\{g_2\left((\nabla\Psi_*)(\tilde{U}_2,\eta\tilde{V}_2) - (\nabla\Psi_*)(\tilde{V}_2,\eta\tilde{U}_2),\Psi_*\eta\mathfrak{B}\tilde{Z}\right)\right\} \\ = \; & g_1(\mathcal{T}_{\tilde{U}_2}\eta\xi\tilde{V}_2 - \mathcal{T}_{\tilde{V}_2}\eta\xi\tilde{U}_2,\tilde{Z}) + g_1(\mathcal{T}_{\tilde{U}_2}\eta\tilde{V}_2 - \mathcal{T}_{\tilde{V}_2}\eta\tilde{U}_2, F\mathfrak{A}\tilde{Z} + \xi\mathfrak{B}\tilde{Z}) \\ & + \frac{1}{\lambda^2}\{g_2(\nabla^{\Psi}_{\tilde{U}_2}\Psi_*\eta\tilde{V}_2 - \nabla^{\Psi}_{\tilde{V}_2}\Psi_*\eta\tilde{U}_2, \Psi_*\eta\mathfrak{B}\tilde{Z})\}. \end{aligned}$$

for any $\tilde{U}_2, \tilde{V}_2 \in \Gamma(\mathfrak{D}^{\theta_2})$ and $\tilde{Z} \in \Gamma(\mathfrak{D}^{\mathfrak{T}} \oplus \mathfrak{D}^{\theta_\perp})$.

Proof. By using equations (2.2), (2.3), (2.13) and (3.3), we have

$$\begin{aligned} g_1([\tilde{U}_2,\tilde{V}_2],\tilde{Z}) = \; & -g_1(\nabla_{\tilde{V}_2}\xi^2\tilde{U}_2,\tilde{Z}) - g_1(\nabla_{\tilde{V}_2}\eta\xi\tilde{U}_2,\tilde{Z}) + g_1(\nabla_{\tilde{U}_2}\xi^2\tilde{V}_2,\tilde{Z}) \\ & +g_1(\nabla_{\tilde{U}_2}\eta\xi\tilde{V}_2,\tilde{Z}) + g_1(\nabla_{\tilde{U}_2}\eta\tilde{V}_2 - \nabla_{\tilde{V}_2}\eta\tilde{U}_2, F\tilde{Z}), \end{aligned}$$

for any $\widetilde{U}_2, \widetilde{V}_2 \in \Gamma(\mathfrak{D}^{\theta_2})$ and $\tilde{Z} \in \Gamma(\mathfrak{D}^{\mathfrak{T}} \oplus \mathfrak{D}^{\theta_1})$. From equations (2.8) and Lemma 3.3 , we get

$$\begin{aligned}\sin^2\theta_2 g_1([\widetilde{U}_2,\widetilde{V}_2],\tilde{Z}) = \quad & g_1(\mathcal{T}_{\widetilde{U}_2}\eta\widetilde{V}_2 - \mathcal{T}_{\widetilde{V}_2}\eta\widetilde{U}_2, F\mathfrak{A}\tilde{Z} + \xi\mathfrak{B}\tilde{Z}) \\ & + g_1(\mathcal{H}\nabla_{\widetilde{U}_2}\eta\widetilde{V}_2 - \mathcal{H}\nabla_{\widetilde{V}_2}\eta\widetilde{U}_2, \eta\mathfrak{B}\tilde{Z}) \\ & + g_1(\mathcal{T}_{\widetilde{U}_2}\eta\xi\widetilde{V}_2 - \mathcal{T}_{\widetilde{V}_2}\eta\xi\widetilde{U}_2, \tilde{Z}).\end{aligned}$$

Since Ψ is CQBS submersion, using conformality condition with equation (2.14), we finally get

$$\begin{aligned}\sin^2\theta_2 g_1([\widetilde{U}_2,\widetilde{V}_2],\tilde{Z}) = \quad & \frac{1}{\lambda^2}\left\{g_2\left((\nabla\Psi_*)(\widetilde{U}_2,\eta\widetilde{V}_2) - (\nabla\Psi_*)(\widetilde{V}_2,\eta\widetilde{U}_2), \Psi_*\eta\mathfrak{B}\tilde{Z}\right)\right\} \\ & + \frac{1}{\lambda^2}\{g_2(\nabla^{\Psi}_{\widetilde{U}_2}\Psi_*\eta\widetilde{V}_2 - \nabla^{\Psi}_{\widetilde{V}_2}\Psi_*\eta\widetilde{U}_2, \Psi_*\eta\mathfrak{B}\tilde{Z})\} \\ & + g_1(\mathcal{T}_{\widetilde{U}_2}\eta\widetilde{V}_2 - \mathcal{T}_{\widetilde{V}_2}\eta\widetilde{U}_2, F\mathfrak{A}\tilde{Z} + \xi\mathfrak{B}\tilde{Z}) \\ & + g_1(\mathcal{T}_{\widetilde{U}_2}\eta\xi\widetilde{V}_2 - \mathcal{T}_{\widetilde{V}_2}\eta\xi\widetilde{U}_2, \tilde{Z})\end{aligned}$$

This completes the proof of the theorem.

Since, the invariant distribution is mutually orthogonal to the slant distributions in accordance with the concept of CQBS-submersion, this led us to investigate the necessary and sufficient condition for the invariant distribution to be integrable.

Theorem 4.3. Let $\Psi: (\bar{Q}_1, g_1, F) \to (\bar{Q}_2, g_2)$ be a CQBS submersion, where $(\bar{Q}_1, g_1, F)$ a LPR manifold and $(\bar{Q}_2, g_2)$ a Riemannian manifold. Then the invariant distribution $\mathfrak{D}^{\mathfrak{T}}$ is integrable if and only if

$$\begin{aligned} & g_1(\mathcal{T}_{\widetilde{U}}\xi\mathfrak{A}\widetilde{V} - \mathcal{T}_{\widetilde{V}}\xi\mathfrak{A}\widetilde{U}, \eta\mathfrak{B}\tilde{Z} + \eta\mathfrak{C}\widetilde{W}) \\ & - g_1(\mathcal{V}\nabla_{\widetilde{U}}\xi\mathfrak{A}\widetilde{V} - \mathcal{V}\nabla_{\widetilde{V}}\xi\mathfrak{A}\widetilde{U}, \xi\mathfrak{B}\tilde{Z} + \xi\mathfrak{C}\tilde{Z}) = 0,\end{aligned} \tag{4.2}$$

for any $\widetilde{U}, \tilde{V} \in \Gamma(\mathfrak{D}^{\mathfrak{T}})$ and $\tilde{Z} \in \Gamma(\mathfrak{D}^{\theta_1} \oplus \mathfrak{D}^{\theta_2})$.

Proof. For all $\tilde{U}, \tilde{V} \in \Gamma(\mathfrak{D}^{\mathfrak{T}})$ and $\tilde{Z} \in \Gamma(\mathfrak{D}^{\theta_1} \oplus \mathfrak{D}^{\theta_2})$ with using equations (2.2), (2.13), (2.7) and decomposition (3.2), we have

$$g_1([\widetilde{U},\tilde{V}],\tilde{Z}) = g_1(\nabla_{\widetilde{U}}\xi\mathfrak{A}\tilde{V}, F\mathfrak{B}\tilde{Z} + F\mathfrak{C}\tilde{Z}) - g_1(\nabla_{\tilde{V}}\xi\mathfrak{A}\widetilde{U}, F\mathfrak{B}\tilde{Z} + F\mathfrak{C}\tilde{Z}).$$

By using equation (3.3), we finally have

$$g_1([\widetilde{U},\widetilde{V}],\widetilde{Z}) = g_1\big(\mathcal{T}_{\widetilde{U}}\xi\mathfrak{A}\widetilde{V} - \mathcal{T}_{\widetilde{V}}\xi\mathfrak{A}\widetilde{U}, \eta\mathfrak{B}\widetilde{Z} + \eta\mathfrak{C}\widetilde{Z}\big) + g_1\big(\mathcal{V}\nabla_{\widetilde{U}}\xi\mathfrak{A}\widetilde{V} - \mathcal{V}\mathcal{A}_{\widetilde{V}}\xi\mathfrak{A}\widetilde{U}, \xi\mathfrak{B}\widetilde{Z} + \xi\mathfrak{C}\widetilde{Z}\big).$$

This completes the proof of theorem.

After discussing the prerequisites for distribution's integrability, it is time to examine the necessary and sufficient conditions that must also exists in order for distributions to be totally geodesic. We begin by looking at the condition of totally geodesicness for invariant distribution.

Theorem 4.4. Let $\Psi: (\bar{Q}_1, g_1, F) \to (\bar{Q}_2, g_2)$ be a CQBS submersion, where $(\bar{Q}_1, g_1, F)$ a LPR manifold and $(\bar{Q}_2, g_2)$ a Riemannian manifold. Then invariant distribution $\mathfrak{D}^{\mathfrak{T}}$ defines totally geodesic foliation on $\bar{Q}_1$ if and only if

(i) $\lambda^{-2} g_2\left((\nabla\Psi_*)(\widetilde{U}, F\widetilde{V}), \Psi_*\eta\widetilde{Z}\right) = g_1\big(\mathcal{V}\nabla_{\widetilde{U}}F\widetilde{V}, \xi\widetilde{Z}\big)$

(ii) $\lambda^{-2} g_2\left((\nabla\Psi_*)(\widetilde{U}, F\widetilde{V}), \Psi_* L\widetilde{X}\right) = g_1\big(\mathcal{V}\nabla_{\widetilde{U}}F\widetilde{V}, P\widetilde{X}\big)$,

for any $\widetilde{U},\widetilde{V} \in \Gamma(\mathfrak{D}^{\mathfrak{T}})$ and $\widetilde{Z} \in \Gamma\big(\mathfrak{D}^{\theta_1} \oplus \mathfrak{D}^{\theta_2}\big)$.

Proof. For any $\widetilde{U},\widetilde{V} \in \Gamma(\mathfrak{D}^{\mathfrak{T}})$ and $\widetilde{Z} \in \Gamma\big(\mathfrak{D}^{\theta_1} \oplus \mathfrak{D}^{\theta_2}\big)$ with using equations (2.2), (2.3), (2.13) and (3.3), we may write

$$g_1\big(\nabla_{\widetilde{U}}\widetilde{V}, \widetilde{Z}\big) = g_1\big(\mathcal{V}\nabla_{\widetilde{U}}F\widetilde{V}, \xi\widetilde{Z}\big) + g_1\big(\mathcal{T}_{\widetilde{U}}F\widetilde{V}, \eta\widetilde{Z}\big).$$

By using the conformality of Ψ with equation (2.14), we get

$$g_1\big(\nabla_{\widetilde{U}}\widetilde{V}, \widetilde{Z}\big) = g_1\big(\mathcal{V}\nabla_{\widetilde{U}}F\widetilde{V}, \xi\widetilde{Z}\big) - \lambda^{-2} g_2\left((\nabla\Psi_*)(\widetilde{U}, F\widetilde{V}), \Psi_*\eta\widetilde{Z}\right).$$

On the other hand, using equations (2.2), (2.3) and (2.13) with conformality of Ψ, we finally have

$$g_1(\nabla_{\tilde{U}}\tilde{V},\tilde{X}) = g_1(\mathcal{V}\nabla_{\tilde{U}}F\tilde{V},P\tilde{X}) - \lambda^{-2}g_2\left((\nabla\Psi_*)(\tilde{U},F\tilde{V}),\Psi_*L\tilde{X}\right).$$

This completes the proof of the theorem.

In similar way, we can discuss the geometry of leaves of slant distribution $\mathfrak{D}^{\theta_1}$ as follows:

Theorem 4.5. Let Ψ be a CQBS submersion from $LPR(\bar{Q}_1, g_1, F)$ onto a Riemannian manifold $(\bar{Q}_2, g_2)$. Then slant distribution $\mathfrak{D}^{\theta_1}$ defines totally geodesic foliation on $\bar{Q}_1$ if and only if

$$\begin{aligned}&\frac{1}{\lambda^2}g_2\left((\nabla\Psi_*)(\tilde{Z},\eta\mathfrak{B}\widetilde{W}),\Psi_*\eta\mathfrak{C}\widetilde{U}\right) - \frac{1}{\lambda^2}g_2(\nabla^{\Psi}_{\tilde{Z}}\Psi_*\eta\mathfrak{B}\widetilde{W},\Psi_*\eta\mathfrak{C}\widetilde{U})\\ &= g_1(\mathcal{T}_{\tilde{Z}}\eta\xi\mathfrak{B}\widetilde{W},\widetilde{U}) - g_1(\mathcal{T}_{\tilde{Z}}\eta\mathfrak{B}\widetilde{W},F\mathfrak{A}\widetilde{U})\\ &\quad - g_1(\mathcal{T}_{\tilde{Z}}\eta\mathfrak{B}\widetilde{W},\xi\mathfrak{C}\widetilde{U}) - \cos^2\theta_1 g_1(\mathcal{V}\nabla_{\tilde{Z}}\mathfrak{B}\widetilde{W},\widetilde{U})\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}&\frac{1}{\lambda^2}g_2\left((\nabla\Psi_*)(\tilde{Z},\eta\xi\mathfrak{B}\widetilde{W}),\Psi_*\tilde{X}\right) + \frac{1}{\lambda^2}g_2\left((\nabla\Psi_*)(\tilde{Z},\eta\xi\mathfrak{B}\widetilde{W}),\Psi_*L\tilde{X}\right)\\ &= \frac{1}{\lambda^2}g_2\left((\nabla\Psi_*)(\tilde{Z},\eta\xi\mathfrak{B}\widetilde{W}),\Psi_*\tilde{X}\right) - \frac{1}{\lambda^2}g_2\left((\nabla\Psi_*)(\tilde{Z},\eta\xi\mathfrak{B}\widetilde{W}),\Psi_*L\tilde{X}\right)\Big)\\ &\quad + \cos^2\theta_1 g_1(\nabla_{\tilde{Z}}\mathfrak{B}\widetilde{W},\tilde{X}) + g_1(\mathcal{T}_{\tilde{Z}}\eta\xi\mathfrak{B}\widetilde{W},P\tilde{X}),\end{aligned} \tag{4.4}$$

for any $\tilde{Z},\widetilde{W} \in \Gamma(\mathfrak{D}^{\theta_1}), \widetilde{U} \in \Gamma(\mathfrak{D}^{\mathfrak{T}} \oplus \mathfrak{D}^{\theta_2})$ and $\tilde{X} \in \Gamma(\ker\Psi_*)^{\perp}$.

Proof. By using equations (2.2), (2.3), (2.13) and (3.3), we get

$$g_1(\nabla_{\tilde{Z}}\widetilde{W},\widetilde{U}) = g_1(\nabla_{\tilde{Z}}\eta\mathfrak{B}\widetilde{W},F(\mathfrak{A}\widetilde{U}+\mathfrak{C}\widetilde{U})) + g_1(F\nabla_{\tilde{Z}}\xi\mathfrak{B}\widetilde{W},\widetilde{U}),$$

for $\tilde{Z},\widetilde{W} \in \Gamma(\mathfrak{D}^{\theta_1})$ and $\widetilde{U} \in \Gamma(\mathfrak{D}^{\mathfrak{T}} \oplus \mathfrak{D}^{\theta_2})$. Again using equations (2.2), (2.3), (2.13), (3.3), (2.8) with Lemma 3.2, we may write

$$\begin{aligned}g_1(\nabla_{\tilde{Z}}\widetilde{W},\widetilde{U}) = \;&\cos^2\theta_1 g_1(\nabla_{\tilde{Z}}\mathfrak{B}\widetilde{W},\widetilde{U}) + g_1(\mathcal{T}_{\tilde{Z}}\eta\xi\mathfrak{B}\widetilde{W},\widetilde{U}) + g_1(\mathcal{T}_{\tilde{Z}}\eta\mathfrak{B}\widetilde{W},F\mathfrak{A}\widetilde{U})\\ &+g_1(\mathcal{T}_{\tilde{Z}}\eta\mathfrak{B}\widetilde{W},\xi\mathfrak{C}\widetilde{U}) + g_1(\mathcal{H}\nabla_{\tilde{Z}}\eta\mathfrak{B}\widetilde{W},\eta\mathfrak{C}\widetilde{U}).\end{aligned}$$

Since, Ψ is conformal, using Lemma 2.1 with equation (2.14), we have

(4.5)

$$g_1(\nabla_{\tilde{Z}}\widetilde{W},\widetilde{U}) = \cos^2\theta_1 g_1(\nabla_{\tilde{Z}}\mathfrak{B}\widetilde{W},\widetilde{U}) + g_1(\mathcal{T}_{\tilde{Z}}\eta\xi\mathfrak{B}\widetilde{W},\widetilde{U}) + g_1(\mathcal{T}_{\tilde{Z}}\eta\mathfrak{B}\widetilde{W},F\mathfrak{A}\widetilde{U}) + g_1(\mathcal{T}_{\tilde{Z}}\eta\mathfrak{B}\widetilde{W},\xi\mathfrak{C}\widetilde{U}) + \frac{1}{\lambda^2}g_2(\nabla^{\Psi}_{\tilde{Z}}\Psi_*\eta\mathfrak{B}\widetilde{W},\Psi_*\eta\mathfrak{C}\widetilde{U}) - \frac{1}{\lambda^2}g_2\left((\nabla\Psi_*)(\tilde{Z},\eta\mathfrak{B}\widetilde{W}),\Psi_*\eta\mathfrak{C}\widetilde{U}\right).$$

On the other hand, for $\tilde{Z},\widetilde{W}\in\Gamma(\mathfrak{D}^{\theta_1})$ and $\tilde{X}\in\Gamma(\ker\Psi_*)^{\perp}$, with using equations (2.2), (2.3), (2.13) and (3.3), we get

$$g_1(\nabla_{\tilde{Z}}\widetilde{W},\tilde{X}) = g_1(\nabla_{\tilde{Z}}\xi\mathfrak{B}\widetilde{W},F\tilde{X}) + g_1(\nabla_{\tilde{Z}}\eta\mathfrak{B}\widetilde{W},F\tilde{X}).$$

From Lemma 3.2 with equations (2.8) and (3.6), the above equation takes the form

$$g_1(\nabla_{\tilde{Z}}\widetilde{W},\tilde{X}) = \cos^2\theta_1 g_1(\nabla_{\tilde{Z}}\tilde{B}\widetilde{W},\tilde{X}) + g_1(\mathcal{H}\nabla_{\tilde{Z}}\eta\xi\mathfrak{B}\widetilde{W},\tilde{X}) + g_1(\mathcal{T}_{\tilde{Z}}\eta\xi\mathfrak{B}\widetilde{W},P\tilde{X}) + g_1(\mathcal{H}\nabla_{\tilde{Z}}\eta\xi\mathfrak{B}\widetilde{W},L\tilde{X}).$$

Since Ψ is conformal and from equation (2.14), we have

$$g_1(\nabla_{\tilde{Z}}\widetilde{W},\tilde{X}) = -\frac{1}{\lambda^2}g_2\left((\nabla\Psi_*)(\tilde{Z},\eta\xi\mathfrak{B}\widetilde{W}),\Psi_*\tilde{X}\right) + \frac{1}{\lambda^2}g_2(\nabla^{\Psi}_{\tilde{Z}}\Psi_*\eta\xi\mathfrak{B}\widetilde{W},\Psi_*\tilde{X}) - \frac{1}{\lambda^2}g_2\left((\nabla\Psi_*)(\tilde{Z},\eta\xi\mathfrak{B}\widetilde{W}),\Psi_*L\tilde{X}\right) + \frac{1}{\lambda^2}g_2(\nabla^{\Psi}_{\tilde{Z}}\Psi_*\eta\xi\mathfrak{B}\widetilde{W},\Psi_*L\tilde{X}) + \cos^2\theta_1 g_1(\nabla_{\tilde{Z}}\widetilde{W},\tilde{X}) + g_1(\mathcal{T}_{\tilde{Z}}\eta\xi\mathfrak{B}\widetilde{W},P\tilde{X}).$$

This completes the proof of theorem.

Theorem 4.6. Let $\Psi:(\bar{Q}_1,g_1,F)\to(\bar{Q}_2,g_2)$ be a CQBS submersion, where $(\bar{Q}_1,g_1,F)$ a LPR manifold and $(\bar{Q}_2,g_2)$ a Riemannian manifold. Then slant distribution $\mathfrak{D}^{\theta_2}$ defines totally geodesic foliation on $\bar{Q}_1$ if and only if

$$\frac{1}{\lambda^2}g_2\left((\nabla\Psi_*)(\tilde{Z},\eta\mathfrak{B}\widetilde{W}),\Psi_*\eta\mathfrak{C}\tilde{V}\right) - \frac{1}{\lambda^2}g_2(\nabla^{\Psi}_{\tilde{Z}}\Psi_*\eta\mathfrak{B}\widetilde{W},\Psi_*\eta\mathfrak{C}\tilde{V}) = g_1(\mathcal{T}_{\tilde{Z}}\eta\xi\mathfrak{B}\widetilde{W},\tilde{V}) - g_1(\mathcal{T}_{\tilde{Z}}\eta\mathfrak{B}\widetilde{W},F\mathfrak{A}\tilde{V}) - g_1(\mathcal{T}_{\tilde{Z}}\eta\mathfrak{B}\widetilde{W},\xi\mathfrak{C}\tilde{V}) - \cos^2\theta_1 g_1(\mathcal{V}\nabla_{\tilde{Z}}\mathfrak{B}\widetilde{W},\tilde{V}), \tag{4.6}$$

and

(4.7)

$$\frac{1}{\lambda^2} g_2\left((\nabla\Psi_*)(\tilde{Z},\eta\xi\mathfrak{B}\widetilde{W}),\Psi_*\tilde{Y}\right)+\frac{1}{\lambda^2} g_2\left((\nabla\Psi_*)(\tilde{Z},\eta\xi\mathfrak{B}\widetilde{W}),\Psi_*L\tilde{Y}\right)$$
$$=\frac{1}{\lambda^2} g_2\left((\nabla\Psi_*)(\tilde{Z},\eta\xi\mathfrak{B}\widetilde{W}),\Psi_*\tilde{Y}\right)-\frac{1}{\lambda^2} g_2\left((\nabla\Psi_*)(\tilde{Z},\eta\xi\mathfrak{B}\widetilde{W}),\Psi_*L\tilde{Y}\right)\Big)$$
$$+\cos^2\theta_2 g_1\left(\nabla_{\tilde{Z}}\mathfrak{B}\widetilde{W},\tilde{Y}\right)+g_1\left(\mathcal{T}_{\tilde{Z}}\eta\xi\mathfrak{B}\widetilde{W},P\tilde{Y}\right),$$

for any $\tilde{Z},\widetilde{W}\in\Gamma(\mathfrak{D}^{\theta_2}),\tilde{V}\in\Gamma(\mathfrak{D}^{\mathfrak{T}}\oplus\mathfrak{D}^{\theta_1})$ and $\tilde{Y}\in\Gamma(\ker\Psi_*)^{\perp}$.

Proof. The proof of above theorem is similar to the proof of Theorem 4.5.

Since, Ψ is CQBS-submersion, its vertical and horizontal distribution are (ker Ψ and $(\ker_*)^{\perp}$, respectively. Now, we examine the necessary and sufficient conditions under which distributions defines totally geodesic foliation on $\bar{Q}_1$. With regards to the totally geodesicness of horizontal distribution, we have

Theorem 4.7. Let Ψ be a CQBS submersion from LPR manifold $(\bar{Q}_1,g_1,F)$ onto a Riemannian manifold $(\bar{Q}_2,g_2)$. Then $(\ker\Psi_*)^{\perp}$ defines totally geodesic

foliation on $\bar{Q}_1$ if and only if

$$\begin{aligned}
&\frac{1}{\lambda^2}\left\{g_2\left(\nabla_{\tilde{X}}^{\Psi}\Psi_*\eta\mathfrak{B}\tilde{Y}+\nabla_{\tilde{X}}^{\Psi}\Psi_*\eta\mathfrak{C}\tilde{Y},\Psi_*\eta\tilde{Z}\right)\right\}\\
&=g_1\left(\mathcal{A}_{\tilde{X}}\eta\xi\mathfrak{B}\tilde{Y}+\mathcal{A}_{\tilde{X}}\eta\xi\mathfrak{C}\tilde{Y}+\mathcal{A}_{\tilde{X}}\eta\xi\mathfrak{A}\tilde{Y}+\mathcal{V}\nabla_{\tilde{X}}\mathfrak{A}\tilde{Y},\tilde{Z}\right)\\
&+\cos^2\theta_1 g_1\left(\mathcal{V}\nabla_{\tilde{X}}\mathfrak{B}\tilde{Y},\tilde{Z}\right)+\cos^2\theta_2 g_1\left(\mathcal{V}\nabla_{\tilde{X}}\mathfrak{C}\tilde{Y},\tilde{Z}\right)\\
&+g_1(\eta\mathfrak{B}\tilde{Y},\eta\tilde{Z})g_1(\tilde{X},\operatorname{grad}\ln\lambda)+g_1(\tilde{X},\eta\tilde{Z})g_1(\eta\mathfrak{B}\tilde{Y},\operatorname{grad}\ln\lambda)\\
&-g_1(\tilde{X},\eta\mathfrak{B}\tilde{Y})g_1(\eta\tilde{Z},\operatorname{grad}\ln\lambda)+g_1(\eta\mathfrak{C}\tilde{Y},\eta\tilde{Z})g_1(\tilde{X},\operatorname{grad}\ln\lambda)\\
&+g_1(\tilde{X},\eta\tilde{Z})g_1(\eta\mathfrak{C}\tilde{Y},\operatorname{grad}\ln\lambda)-g_1(\tilde{X},\eta\mathfrak{C}\tilde{Y})g_1(\eta\tilde{Z},\operatorname{grad}\ln\lambda),
\end{aligned} \tag{4.8}$$

for any $\tilde{X},\tilde{Y}\in\Gamma(\ker\Psi_*)^{\perp}$ and $\tilde{Z}\in\Gamma(\ker\Psi_*)$.

Proof. For any $\tilde{X},\tilde{Y}\in\Gamma(\ker\Psi_*)^{\perp}$ and $\tilde{Z}\in\Gamma(\ker\Psi_*)$ with using equations (2.2), (2.3) and (2.13) with decomposition (3.2), we get

$$g_1(\nabla_{\tilde{X}}\tilde{Y},\tilde{Z}) = g_1(\nabla_{\tilde{X}}F(\mathfrak{A}\tilde{Y}),F\tilde{Z}) + g_1(\nabla_{\tilde{X}}F(\mathfrak{B}\tilde{Y}),F\tilde{Z}) + g_1(\nabla_{\tilde{X}}F(\mathfrak{C}\tilde{Y}),F\tilde{Z}).$$

From equations (3.3) and (2.9) with Lemma 3.2, we have

$$\begin{aligned} g_1(\nabla_{\tilde{X}}\tilde{Y},\tilde{Z}) = \quad & g_1(\mathcal{V}\nabla_{\tilde{X}}\mathfrak{A}\tilde{Y},\tilde{Z}) + \cos^2\theta_1 g_1(\nabla_{\tilde{X}}\mathfrak{B}\tilde{Y},\tilde{Z}) + \cos^2\theta_2 g_1(\nabla_{\tilde{X}}\mathfrak{C}\tilde{Y},\tilde{Z}) \\ & + g_1(\nabla_{\tilde{X}}\eta\xi\mathfrak{B}\tilde{Y},\tilde{Z}) + g_1(\nabla_{\tilde{X}}\eta\mathfrak{B}\tilde{Y},F\tilde{Z}) + g_1(\nabla_{\tilde{X}}\eta\xi\mathfrak{C}\tilde{Y},\tilde{Z}) \\ & + g_1(\nabla_{\tilde{X}}\eta\mathfrak{C}\tilde{Y},F\tilde{Z}) + g_1(\nabla_{\tilde{X}}\eta\xi\mathfrak{A}\tilde{Y},\tilde{Z}) \end{aligned}$$

By using the equations (3.3) and (2.10), we get

$$\begin{aligned} & g_1(\nabla_{\tilde{X}}\tilde{Y},\tilde{Z}) = g_1(\mathcal{V}\nabla_{\tilde{X}}\mathfrak{A}\tilde{Y} + \cos^2\theta_1\mathcal{V}\nabla_{\tilde{X}}\mathfrak{B}\tilde{Y} + \cos^2\theta_2\mathcal{V}\nabla_{\tilde{X}}\mathfrak{C}\tilde{Y},\tilde{Z}) \\ & + g_1(\mathcal{A}_{\tilde{X}}\eta\xi\mathfrak{A}\tilde{Y} + \mathcal{A}_{\tilde{X}}\eta\xi\mathfrak{B}\tilde{Y} + \mathcal{A}_{\tilde{X}}\eta\xi\mathfrak{C}\tilde{Y},\tilde{Z}) \\ & + g_1(\mathcal{H}\nabla_{\tilde{X}}\eta\mathfrak{B}\tilde{Y} + \mathcal{H}\nabla_{\tilde{X}}\eta\mathfrak{C}\tilde{Y},\eta\tilde{Z}) \\ & + g_1(\mathcal{A}_{\tilde{X}}\eta\mathfrak{B}\tilde{Y} + \mathcal{A}_{\tilde{X}}\eta\mathfrak{C}\tilde{Y},\xi\tilde{Z}). \end{aligned}$$

From formula (2.14), we yields that

$$\begin{aligned} & g_1(\nabla_{\tilde{X}}\tilde{Y},\tilde{Z}) = g_1(\mathcal{V}\nabla_{\tilde{X}}\mathfrak{A}\tilde{Y} + \cos^2\theta_1\mathcal{V}\nabla_{\tilde{X}}\mathfrak{B}\tilde{Y} + \cos^2\theta_2\mathcal{V}\nabla_{\tilde{X}}\mathfrak{C}\tilde{Y},\tilde{Z}) \\ & + g_1(\mathcal{A}_{\tilde{X}}\eta\xi\mathfrak{A}\tilde{Y} + \mathcal{A}_{\tilde{X}}\eta\xi\mathfrak{B}\tilde{Y} + \mathcal{A}_{\tilde{X}}\eta\xi\mathfrak{C}\tilde{Y},\tilde{Z}) \\ & + \frac{1}{\lambda^2}\{g_2(\nabla^{\Psi}_{\tilde{X}}\Psi_*\eta\mathfrak{B}\tilde{Y} + \nabla^{\Psi}_{\tilde{X}}\Psi_*\eta\mathfrak{C}\tilde{Y},\Psi_*\eta\tilde{Z})\} \\ & - \frac{1}{\lambda^2}\left\{g_2\left((\nabla\Psi_*)(\tilde{X},\eta\mathfrak{B}\tilde{Y}) + (\nabla\Psi_*)(\tilde{X},\eta\mathfrak{C}\tilde{Y}),\Psi_*\eta\tilde{Z}\right)\right\} \end{aligned}$$

Since Ψ is conformal submersion, then we finally get

$$g_1(\nabla_{\tilde{X}}\tilde{Y},\tilde{Z}) = -g_1(\eta\mathfrak{B}\tilde{Y},\eta\tilde{Z})g_1(\tilde{X},\text{grad}\ln\lambda) - g_1(\tilde{X},\eta\tilde{Z})g_1(\eta\mathfrak{B}\tilde{Y},\text{grad}\ln\lambda)$$
$$+g_1(\tilde{X},\eta\mathfrak{B}\tilde{Y})g_1(\eta\tilde{Z},\text{grad}\ln\lambda) - g_1(\eta\mathfrak{C}\tilde{Y},\eta\tilde{Z})g_1(\tilde{X},\text{grad}\ln\lambda)$$
$$-g_1(\tilde{X},\eta\tilde{Z})g_1(\eta\mathfrak{C}\tilde{Y},grad\ln\lambda) + g_1(\tilde{X},\eta\mathfrak{C}\tilde{Y})g_1(\eta\tilde{Z},\text{grad}\ln\lambda)$$
$$+g_1(\mathcal{V}\nabla_{\tilde{X}}\mathfrak{A}\tilde{Y} + \cos^2\theta_1\mathcal{V}\nabla_{\tilde{X}}\mathfrak{B}\tilde{Y} + \cos^2\theta_2\mathcal{V}\nabla_{\tilde{X}}\mathfrak{C}\tilde{Y},\tilde{Z})$$
$$+g_1(\mathcal{A}_{\tilde{X}}\eta\xi\mathfrak{A}\tilde{Y} + \mathcal{A}_{\tilde{X}}\eta\xi\mathfrak{B}\tilde{Y} + \mathcal{A}_{\tilde{X}}\eta\xi\mathfrak{C}\tilde{Y},\tilde{Z})$$
$$+\frac{1}{\lambda^2}\{g_2(\nabla^{\Psi}_{\tilde{X}}\Psi_*\eta\mathfrak{B}\tilde{Y} + \nabla^{\Psi}_{\tilde{X}}\Psi_*\eta\mathfrak{C}\tilde{Y},\Psi_*\eta\tilde{Z})\}.$$

This completes the proof of theorem.

We can now talk about the geometry of leaves of horizontal distribution. The following theorem presents the necessary and sufficient condition under which vertical distribution defines totally geodesic foliation on $\bar{Q}_1$.

Theorem 4.8. Let $\Psi: (\bar{Q}_1, g_1, F) \to (\bar{Q}_2, g_2)$ be a CQBS submersion, where $(\bar{Q}_1, g_1, F)$ a LPR manifold and $(\bar{Q}_2, g_2)$ a Riemannian manifold. Then $(\ker\Psi_*)$ defines totally geodesic foliation on $\bar{Q}_1$ if and only if

$$\begin{aligned}\frac{1}{\lambda^2}\ &\{g_2(\nabla^{\Psi}_{\tilde{U}}\Psi_*\eta\xi\mathfrak{B}\tilde{V} + \nabla^{\Psi}_{\tilde{U}}\Psi_*\eta\xi\mathfrak{C}\tilde{V},\Psi_*\tilde{X})\}\\ =\ &g_1(\mathcal{T}_{\tilde{U}}\mathfrak{A}\tilde{V} + \cos^2\theta_1\mathcal{T}_{\tilde{U}}\mathfrak{B}\tilde{V} + \cos^2\theta_2\mathcal{T}_{\tilde{U}}\mathfrak{C}\tilde{V}) + g_1(\mathcal{T}_{\tilde{U}}\eta\tilde{V},P\tilde{X})\\ &-\frac{1}{\lambda^2}\left\{g_2\left((\nabla\Psi_*)(\tilde{U},\eta\xi\mathfrak{B}\tilde{V}) + (\nabla\Psi_*)(\tilde{U},\eta\xi\mathfrak{C}\tilde{V}),\Psi_*\tilde{X}\right)\right\}\\ &+\frac{1}{\lambda^2}\{g_2(\nabla^{\Psi}_{\tilde{U}}\Psi_*\eta\tilde{V} - (\nabla\Psi_*)(\tilde{U},\eta\tilde{V}),\Psi_*L\tilde{X})\},\end{aligned} \tag{4.9}$$

for any $\tilde{U},\tilde{V} \in \Gamma(\ker\Psi_*)$ and $\tilde{X} \in \Gamma(\ker\Psi_*)^{\perp}$.

Proof. For any $\tilde{U},\tilde{V} \in \Gamma(\ker\Psi_*)$ and $\tilde{X} \in \Gamma(\ker\Psi_*)^{\perp}$ with using equations (2.2), (2.3), (2.13) with decomposition (3.2), we get

$$g_1(\nabla_{\tilde{U}}\tilde{V},\tilde{X}) = g_1(\nabla_{\tilde{U}}F\mathfrak{A}\tilde{V},F\tilde{X}) + g_1(\nabla_{\tilde{U}}F\mathfrak{B}\tilde{V},F\tilde{X}) + g_1(\nabla_{\tilde{U}}F\mathfrak{C}\tilde{V},F\tilde{X}).$$

By using equations (3.3) with Lemma 3.2 and Lemma 3.3, we have

$$\begin{aligned} g_1(\nabla_{\tilde{U}}\tilde{V},\tilde{X}) = \; & g_1(\nabla_{\tilde{U}}\mathfrak{A}\tilde{V},\tilde{X}) + \cos^2\theta_1 g_1(\nabla_{\tilde{U}}\mathfrak{B}\tilde{V},\tilde{X}) + \cos^2\theta_2 g_1(\nabla_{\tilde{U}}\mathfrak{C}\tilde{V},\tilde{X}) \\ & + g_1(\nabla_{\tilde{U}}\eta\mathfrak{B}\tilde{V},F\tilde{X}) + g_1(\nabla_{\tilde{U}}\eta\xi\mathfrak{B}\tilde{V},\tilde{X}) + g_1(\nabla_{\tilde{U}}\eta\xi\mathfrak{C}\tilde{V},\tilde{X}) \\ & + g_1(\nabla_{\tilde{U}}\eta\mathfrak{C}\tilde{V},F\tilde{X}). \end{aligned}$$

From equations (2.7), (2.8) and (3.6), we may yields

$$\begin{aligned} g_1(\nabla_{\tilde{U}}\tilde{V},\tilde{X}) = \; & g_1(\mathcal{T}_{\tilde{U}}\mathfrak{A}\tilde{V} + \cos^2\theta_1\mathcal{T}_{\tilde{U}}\mathfrak{B}\tilde{V} + \cos^2\theta_2\mathcal{T}_{\tilde{U}}\mathfrak{C}\tilde{V},\tilde{X}) \\ & + g_1(\mathcal{H}\nabla_{\tilde{U}}\eta\xi\mathfrak{B}\tilde{V} + \mathcal{H}\nabla_{\tilde{U}}\eta\xi\mathfrak{C}\tilde{V},\tilde{X}) \\ & + g_1(\mathcal{H}\nabla_{\tilde{U}}\eta\mathfrak{B}\tilde{V} + \mathcal{H}\nabla_{\tilde{U}}\eta\mathfrak{C}\tilde{V},L\tilde{X}) \\ & + g_1(\mathcal{T}_{\tilde{U}}\eta\mathfrak{B}\tilde{V} + \mathcal{T}_{\tilde{U}}\eta\mathfrak{C}\tilde{V},P\tilde{X}) \end{aligned}$$

From decomposition (3.2), the above equation takes the form

$$\begin{aligned} g_1(\nabla_{\tilde{U}}\tilde{V},\tilde{X}) = \; & g_1(\mathcal{T}_{\tilde{U}}\mathfrak{A}\tilde{V} + \cos^2\theta_1\mathcal{T}_{\tilde{U}}\mathfrak{B}\tilde{V} + \cos^2\theta_2\mathcal{T}_{\tilde{U}}\mathfrak{C}\tilde{V},\tilde{X}) + g_1(\mathcal{T}_{\tilde{U}}\eta\tilde{V},P\tilde{X}) \\ & + g_1(\mathcal{H}\nabla_{\tilde{U}}\eta\xi\mathfrak{B}\tilde{V} + \mathcal{H}\nabla_{\tilde{U}}\eta\xi\mathfrak{C}\tilde{V},\tilde{X}) + g_1(\mathcal{H}\nabla_{\tilde{U}}\eta\tilde{V},L\tilde{X}). \end{aligned}$$

Using the conformality of Ψ with equation (2.14), we have

$$\begin{aligned} g_1(\nabla_{\tilde{U}}\tilde{V},\tilde{X}) = \; & g_1(\mathcal{T}_{\tilde{U}}\mathfrak{A}\tilde{V} + \cos^2\theta_1\mathcal{T}_{\tilde{U}}\mathfrak{B}\tilde{V} + \cos^2\theta_2\mathcal{T}_{\tilde{U}}\mathfrak{C}\tilde{V},\tilde{X}) + g_1(\mathcal{T}_{\tilde{U}}\eta\tilde{V},P\tilde{X}) \\ & - \frac{1}{\lambda^2}\left\{g_2\left((\nabla\Psi_*)(\tilde{U},\eta\xi\mathfrak{B}\tilde{V}) + (\nabla\Psi_*)(\tilde{U},\eta\xi\mathfrak{C}\tilde{V}),\Psi_*\tilde{X}\right)\right\} \\ & - \frac{1}{\lambda^2}\left\{g_2(\nabla^{\Psi}_{\tilde{U}}\Psi_*\eta\xi\mathfrak{B}\tilde{V} + \nabla^{\Psi}_{\tilde{U}}\Psi_*\eta\xi\mathfrak{C}\tilde{V},\Psi_*\tilde{X})\right\} \\ & + \frac{1}{\lambda^2}\left\{g_2(\nabla^{\Psi}_{\tilde{U}}\Psi_*\eta\tilde{V} - (\nabla\Psi_*)(\tilde{U},\eta\tilde{V}),\Psi_*L\tilde{X})\right\}. \end{aligned}$$

This completes the proof of the theorem.

We now have some necessary and sufficient conditions for a CQBS submersion $\Psi: \bar{Q}_1 \to \bar{Q}_2$ to be totally geodesic map. In this regard, we are presenting the following theorem.

Theorem 4.9. Let Ψ be a CQBS submersion from LPR manifold $(\bar{Q}_1, g_1, F)$ onto a Riemannian manifold $(\bar{Q}_2, g_2)$. Then $\Psi: (\bar{Q}_1, g_1, F) \to (\bar{Q}_1, g_2)$ is totally geodesic map if and only if

(4.10)

(4.11)

$$\Psi_*\{\cos^2\theta_1\nabla_{\tilde{U}}\mathfrak{B}\tilde{V}+\cos^2\theta_2\nabla_{\tilde{U}}\mathfrak{C}\tilde{V}+\nabla_{\tilde{U}}\eta\xi\mathfrak{B}\tilde{V}+\nabla_{\tilde{U}}\eta\xi\mathfrak{C}\tilde{V}\}$$
$$=\Psi_*\{L(-\mathcal{H}\nabla_{\tilde{U}}\eta\mathfrak{B}\tilde{V}-\mathcal{H}\nabla_{\tilde{U}}\eta\mathfrak{C}\tilde{V}-\mathcal{T}_{\tilde{U}}\xi\mathfrak{A}\tilde{V})\}$$
$$-\Psi_*\{\eta(\mathcal{T}_{\tilde{U}}\eta\mathfrak{B}\tilde{V}+\mathcal{T}_{\tilde{U}}\eta\mathfrak{C}\tilde{V}+\mathcal{V}\nabla_{\tilde{U}}\xi\mathfrak{A}\tilde{V})\},$$
$$\Psi_*\{\cos^2\theta_1\nabla_{\tilde{X}}\mathfrak{B}\tilde{U}+\cos^2\theta_2\nabla_{\tilde{X}}\mathfrak{C}\tilde{U}+\nabla_{\tilde{X}}\eta\xi\mathfrak{B}\tilde{U}+\nabla_{\tilde{X}}\eta\xi\mathfrak{C}\tilde{U}\}$$
$$=-\Psi_*\{L(\mathcal{A}_{\tilde{X}}\xi\mathfrak{A}\tilde{U}+\mathcal{H}\nabla_{\tilde{X}}\eta\mathfrak{B}\tilde{U}+\mathcal{H}\nabla_{\tilde{X}}\eta\mathfrak{C}\tilde{U})\}$$
$$-\Psi_*\{\eta(\mathcal{V}\nabla_{\tilde{X}}\xi\mathfrak{A}\tilde{U}+\mathcal{A}_{\tilde{X}}\eta\mathfrak{B}\tilde{U}+\mathcal{A}_{\tilde{X}}\eta\tilde{C}\tilde{U})\}$$

for any $\tilde{U},\tilde{V}\in\Gamma(\ker\Psi_*)$ and $\tilde{X},\tilde{Y}\in\Gamma(\ker\Psi_*)^{\perp}$.

Proof. Now, using equations (2.14), (2.3), (2.13) and (2.1).

$$(\nabla\Psi_*)(\tilde{U},\tilde{V})=-\Psi_*(F\nabla_{\tilde{U}}F\tilde{V}),$$

for any $\tilde{U},\tilde{V}\in\Gamma(\ker\Psi_*)$. From decomposition (3.2) and equation (3.3), we may write

$$\begin{aligned}(\nabla\Psi_*)(\tilde{U},\tilde{V})= \quad &\Psi_*\{-F\nabla_{\tilde{U}}\xi\mathfrak{A}\tilde{V}-F\nabla_{\tilde{U}}\xi\mathfrak{B}\tilde{V}-F\nabla_{\tilde{U}}\eta\mathfrak{B}\tilde{V}\\ &-F\nabla_{\tilde{U}}\xi\mathfrak{C}\tilde{V}-F\nabla_{\tilde{U}}\eta\mathfrak{C}\tilde{V}\}.\end{aligned}$$

By using equations (2.7) and (2.8), the above equation takes the form

$$\begin{aligned}(\nabla\Psi_*)(\tilde{U},\tilde{V})= \quad &\Psi_*\{-F\mathcal{T}_{\tilde{U}}\xi\mathfrak{A}\tilde{V}-F\mathcal{V}\nabla_{\tilde{U}}\xi\mathfrak{A}\tilde{V}\}-\Psi_*(\nabla_{\tilde{U}}F\xi\mathfrak{B}\tilde{V})\\ &-\Psi_*\{F\mathcal{T}_{\tilde{U}}\eta\mathfrak{B}\tilde{V}+F\mathcal{H}\nabla_{\tilde{U}}\eta\mathfrak{B}\tilde{V}\}-\Psi_*(\nabla_{\tilde{U}}F\xi\mathfrak{C}\tilde{V})\\ &-\Psi_*\{F\mathcal{T}_{\tilde{U}}\eta\mathfrak{C}\tilde{V}+F\mathcal{H}\nabla_{\tilde{U}}\eta\mathfrak{C}\tilde{V}\}.\end{aligned}$$

Since Ψ is conformal submersion, by using Lemma 3.2 and Lemma 3.3 with equation (3.3), we finally get

$$\begin{aligned}(\nabla\Psi_*)(\tilde{U},\tilde{V})= \quad &\Psi*\{L(-\mathcal{H}\nabla_{\tilde{U}}\eta\mathfrak{B}\tilde{V}-\mathcal{H}\nabla_{\tilde{U}}\eta\mathfrak{C}\tilde{V}-\mathcal{T}_{\tilde{U}}\xi\mathfrak{A}\tilde{V})\\ &+\eta(-\mathcal{V}\nabla_{\tilde{U}}\xi\mathfrak{A}\tilde{V}-\mathcal{T}_{\tilde{U}}\xi\mathfrak{B}\tilde{V}-\mathcal{T}_{\tilde{U}}\xi\mathfrak{C}\tilde{V})\}\\ &-\Psi_*\{\cos^2\theta_1\nabla_{\tilde{U}}\mathfrak{B}\tilde{V}+\cos^2\theta_2\nabla_{\tilde{U}}\mathfrak{C}\tilde{V}+\nabla_{\tilde{U}}\eta\xi\mathfrak{B}\tilde{V}+\nabla_{\tilde{U}}\eta\xi\mathfrak{C}\tilde{V}\}\end{aligned}$$

From this, the (i) part of theorem proved. On the other hand, for $\widetilde{U} \in \Gamma(\ker \Psi_*)$ and $\tilde{X} \in \Gamma(\ker \Psi_*)^{\perp}$ with using equations (2.14), (2.3), (2.13) and (2.1), we can write

$$(\nabla\Psi_*)(\tilde{X}, \widetilde{U}) = -\Psi_*\left(F\nabla_{\tilde{X}} F\widetilde{U}\right).$$

By using decomposition (3.2) with equation (3.3), we have

$$(\nabla\Psi_*)(\tilde{X}, \widetilde{U}) = -\Psi_*\left\{F\left(\nabla_{\tilde{X}}\xi\mathfrak{A}\widetilde{U} + \nabla_{\tilde{X}}\xi\mathfrak{B}\widetilde{U} + \nabla_{\tilde{X}}\eta\mathfrak{B}\widetilde{U} + \nabla_{\tilde{X}}\xi\mathfrak{C}\widetilde{U} + \nabla_{\tilde{X}}\eta\mathfrak{C}\widetilde{U}\right)\right\}.$$

By taking account the fact from equations (2.9) and (2.10), we get

$$\begin{aligned}(\nabla\Psi_*)(\tilde{X}, \widetilde{U}) = \quad & -\Psi_*\left\{F\left(\mathcal{A}_{\tilde{X}}\xi\mathfrak{A}\widetilde{U} + \mathcal{V}\nabla_{\tilde{X}}\xi\mathfrak{A}\widetilde{U} + \nabla_{\tilde{X}}F\xi\mathfrak{B}\widetilde{U}\right.\right. \\ & +F\left(\mathcal{H}\nabla_{\tilde{X}}\eta\mathfrak{B}\widetilde{U} + \mathcal{A}_{\tilde{X}}\eta\mathfrak{B}\widetilde{U}\right) + \nabla_{\tilde{X}}F\xi\mathfrak{C}\widetilde{U} \\ & \left.\left.+F\left(\mathcal{H}\nabla_{\tilde{X}}\eta\mathfrak{C}\widetilde{U} + \mathcal{A}_{\tilde{X}}\eta\mathfrak{C}\widetilde{U}\right)\right\}.\end{aligned}$$

Finally, from conformality of Riemannian submersion Ψ and Lemma 3.2, Lemma 3.3 , we can write

$$\begin{aligned}(\nabla\Psi_*)(\tilde{X}, \widetilde{U}) = \quad & -\Psi_*\left\{L\left(\mathcal{A}_{\tilde{X}}\xi\mathfrak{A}\widetilde{U} + \mathcal{H}\nabla_{\tilde{X}}\eta\mathfrak{B}\widetilde{U} + \mathcal{H}\nabla_{\tilde{X}}\eta\mathfrak{C}\widetilde{U}\right)\right\} \\ & -\Psi_*\left\{\eta\left(\mathcal{V}\nabla_{\tilde{X}}\xi\mathfrak{A}\widetilde{U} + \mathcal{A}_{\tilde{X}}\eta\mathfrak{B}\widetilde{U} + \mathfrak{A}_{\tilde{X}}\eta\mathfrak{C}\widetilde{U}\right)\right\} \\ & -\Psi_*\left(\cos^2\theta_1\nabla_{\tilde{X}}\mathfrak{B}\widetilde{U} + \cos^2\theta_2\nabla_{\tilde{X}}\mathfrak{C}\widetilde{U} + \nabla_{\tilde{X}}\eta\xi\mathfrak{B}\widetilde{U} + \nabla_{\tilde{X}}\eta\xi\mathfrak{C}\widetilde{U}\right).\end{aligned}$$

From which we obtain (ii) part of theorem. This completes the proof of theorem.

5 Decomposition Theorems

In this section, we recall the following result from [22] and discuss some decomposition theorems. Let g be a Riemannian metric tensor on the product manifold $M = \bar{Q}_1 \times \bar{Q}_2$ where $\bar{Q}_1$ and $\bar{Q}_2$ are two Riemannian manifold, then the from following conditions, it is easy to understand the concepts of locally product manifold and twisted product manifold.

(i) $M = \bar{Q}_1 \times_\lambda \bar{Q}_2$ is a locally product if and only if $\bar{Q}_1$ and $\bar{Q}_2$ are totally geodesic foliations,

(ii) a warped product $\bar{Q}_1 \times_f \bar{Q}_2$ if and only if $\bar{Q}_1$ is a totally geodesic foliation and $\bar{Q}_2$ is a spherics foliation, i.e., it is umbilic and its mean curvature vector field is parallel,

(iii) $M = \bar{Q}_1 \times_\lambda \bar{Q}_2$ is a twisted product if and only if $\bar{Q}_1$ is a totally geodesic foliation and $\bar{Q}_2$ is a totally umbilic foliation.

The presence of three orthogonal complementary distributions $\mathfrak{D}^{\mathfrak{T}}, \mathfrak{D}^{\theta_1}$, and $\mathfrak{D}^{\theta_2}$, which are integrable and totally geodesic under the conditions that we have stated previously, is ensured by the fact that $\Psi: (\bar{Q}_1, g_1, F) \to (\bar{Q}_2, g_2)$ is CQBS submersion. It makes sense to now look for the conditions in which the total space $\bar{Q}_1$ converts into locally product manifolds or locally twisted product manifolds. In order to explore the geometry of conformal bi-slant submersion Ψ, we are providing here a few decomposition theorems that state that $\bar{Q}_1$ converts into locally product manifolds in a variety of situations.

Theorem 5.1. Let $\Psi: (\bar{Q}_1, g_1, F) \to (\bar{Q}_2, g_2)$ be a CQBS submersion, where $(\bar{Q}_1, g_1, F)$ a LPR manifold and $(\bar{Q}_2, g_2)$ a Riemannian manifold. Then $\bar{Q}_1$ is a locally product manifold if and only if

$$\begin{aligned} \frac{1}{\lambda^2} & \left\{g_2\left(\nabla_{\tilde{U}}^{\Psi} \Psi_* \eta \xi \mathfrak{B} \tilde{V} + \nabla_{\tilde{U}}^{\Psi} \Psi_* \eta \xi \mathfrak{C} \tilde{V}, \Psi_* \tilde{X}\right)\right\} \\ = \; & g_1\left(\mathcal{T}_{\tilde{U}} \mathfrak{A} \tilde{V} + \cos^2\theta_1 \mathcal{T}_{\tilde{U}} \mathfrak{B} \tilde{V} + \cos^2\theta_2 \mathcal{T}_{\tilde{U}} \mathfrak{C} \tilde{V}\right) + g_1\left(\mathcal{T}_{\tilde{U}} \eta \tilde{V}, P\tilde{X}\right) \\ & -\frac{1}{\lambda^2}\left\{g_2\left((\nabla \Psi_*)(\tilde{U}, \eta\xi\mathfrak{B}\tilde{V}) + (\nabla \Psi_*)(\tilde{U}, \eta\xi\mathfrak{C}\tilde{V}), \Psi_* \tilde{X}\right)\right\} \\ & +\frac{1}{\lambda^2}\left\{g_2\left(\nabla_{\tilde{U}}^{\Psi} \Psi_* \eta \tilde{V} - (\nabla \Psi_*)(\tilde{U}, \eta \tilde{V}), \Psi_* L\tilde{X}\right)\right\} \end{aligned} \tag{5.1}$$

and

(5.2)

$$\frac{1}{\lambda^2}\{g_2(\nabla^{\Psi}_{\tilde{X}}\Psi_*\eta\mathfrak{B}\tilde{Y}+\nabla^{\Psi}_{\tilde{X}}\Psi_*\eta\tilde{C}\tilde{Y},\Psi_*\eta\tilde{Z})\}$$
$$=g_1(\mathcal{A}_{\tilde{X}}\eta\xi\mathfrak{B}\tilde{Y}+\mathcal{A}_{\tilde{X}}\eta\xi\mathfrak{C}\tilde{Y}+\mathcal{A}_{\tilde{X}}\eta\xi\mathfrak{A}\tilde{Y},\tilde{Z})$$
$$+g_1(\mathcal{V}\nabla_{\tilde{X}}\mathfrak{A}\tilde{Y}+\cos^2\theta_1\mathcal{V}\nabla_{\tilde{X}}\mathfrak{B}\tilde{Y}+\cos^2\theta_2\mathcal{V}\nabla_{\tilde{X}}\mathfrak{C}\tilde{Y},\tilde{Z})$$
$$+g_1(\eta\mathfrak{B}\tilde{Y},\eta\tilde{Z})g_1(\tilde{X},\operatorname{grad}\ln\lambda)+g_1(\tilde{X},\eta\tilde{Z})g_1(\eta\mathfrak{B}\tilde{Y},\operatorname{grad}\ln\lambda)$$
$$-g_1(\tilde{X},\eta\mathfrak{B}\tilde{Y})g_1(\eta\tilde{Z},\operatorname{grad}\ln\lambda)+g_1(\eta\mathfrak{C}\tilde{Y},\eta\tilde{Z})g_1(\tilde{X},\operatorname{grad}\ln\lambda)$$
$$+g_1(\tilde{X},\eta\tilde{Z})g_1(\eta\mathfrak{C}\tilde{Y},\operatorname{grad}\ln\lambda)-g_1(\tilde{X},\eta\mathfrak{C}\tilde{Y})g_1(\eta\tilde{Z},\operatorname{grad}\ln\lambda),$$

for any $\tilde{U},\tilde{V}\in\Gamma(\ker\Psi_*)$ and $\tilde{X},\tilde{Y}\in\Gamma(\ker\Psi_*)^{\perp}$.

Proof. The proof of this theorem is directly from Theorem 4.7 and Theorem 4.8 .

Since we discussed in the previous theorem, given certain necessary and sufficient conditions, the total space $\bar{Q}_1$ transforms into a locally product manifold. Now, it's intriguing to investigate if there are any circumstances under which the total space $\bar{Q}_1$ could turn into a locally twisted product manifold. The conditions that turn total space $\bar{Q}_1$ into a locally twisted product manifold are found in the following result.

Theorem 5.2. Let Ψ be a CQBS submersion from LPR manifold $(\bar{Q}_1,g_1,F)$ onto a Riemannian manifold $(\bar{Q}_2,g_2)$. Then $\bar{Q}_1$ is locally twisted product of the form $\bar{Q}_{1(\ker\Psi_*)}\times\bar{Q}_{1(\ker\Psi_*)\perp}$ if and only if

$$\frac{1}{\lambda^2}g_2\left((\nabla\Psi_*)(\tilde{U},\eta\tilde{V}),\Psi_*L\tilde{X}\right)=g_1(\nabla_{\tilde{U}}\xi\tilde{V},F\tilde{X})+g_1(\mathcal{T}_{\tilde{U}}\eta\tilde{V},P\tilde{X})+\frac{1}{\lambda^2}g_2(\nabla^{\Psi}_{\tilde{U}}\Psi_*\eta\tilde{V},\Psi_*L\tilde{X}). \tag{5.3}$$

and

$$g_1(\tilde{X},\tilde{Y})H=-P\mathcal{A}_{\tilde{X}}P\tilde{Y}-\xi\nabla_{\tilde{X}}P\tilde{Y}-\xi\mathcal{A}_{\tilde{X}}L\tilde{Y}-F\Psi_*(\nabla^{\Psi}_{\tilde{X}}\Psi_*L\tilde{Y})+\tilde{X}(\ln\lambda)PL\tilde{Y}+L\tilde{Y}(\ln\lambda)P\tilde{X}-P(\operatorname{grad}\ln\lambda)g_1(\tilde{X},L\tilde{Y}). \tag{5.4}$$

where H is a mean curvature vector and for any $\tilde{U},\tilde{V}\in\Gamma(\ker\Psi_*)$ and $\tilde{X}_1,\tilde{X}_2\in\Gamma(\ker\Psi_*)^{\perp}$.

Proof. For any $\tilde{X} \in \Gamma(\ker \Psi_*)^{\perp}$ and $\tilde{U}, \tilde{V} \in \Gamma(\ker \Psi_*)$ and using equations (2.2), (2.3), (2.13), (2.7), (2.8) (3.3) and (3.6), we have

$$g_1(\nabla_{\tilde{U}}\tilde{V}, \tilde{X}) = g_1(\nabla_{\tilde{U}}\xi\tilde{V}, F\tilde{X}) + g_1(\mathcal{T}_{\tilde{U}}\eta\tilde{V}, P\tilde{X}) + g_1(\mathcal{H}\nabla_{\tilde{U}}\eta\tilde{V}, L\tilde{X}).$$

From using formula (2.14) and definition of conformality, the above equation takes place as

$$\begin{aligned} g_1(\nabla_{\tilde{U}}\tilde{V}, \tilde{X}) = \quad & g_1(\nabla_{\tilde{U}}\xi\tilde{V}, F\tilde{X}) + g_1(\mathcal{T}_{\tilde{U}}\eta\tilde{V}, P\tilde{X}) - \frac{1}{\lambda^2} g_2\left((\nabla\Psi_*)(\tilde{U}, \eta\tilde{V}), \Psi_* L\tilde{X}\right) \\ & + \frac{1}{\lambda^2} g_2(\nabla^{\Psi}_{\tilde{U}}\Psi_*\eta\tilde{V}, \Psi_* L\tilde{X}). \end{aligned}$$

It follows that the equation (5.3) satisfies if and only if $\bar{Q}_{1(\ker\Psi_*)}$ is totally geodesic. On the other hand, for $\tilde{U} \in \Gamma(\ker \Psi_*)$ and $\tilde{X}, \tilde{Y} \in \Gamma(\ker \Psi_*)^{\perp}$ with using equations (2.2), (2.13), (2.3) (2.10), (3.3) and (3.6), we get

$$g_1(\nabla_{\tilde{X}}\tilde{Y}, \tilde{U}) = g_1(\nabla_{\tilde{X}}P\tilde{Y}, F\tilde{U}) + g_1(\mathcal{A}_{\tilde{X}}L\tilde{Y}, \xi\tilde{U}) + g_1(\mathcal{H}\nabla_{\tilde{X}}L\tilde{Y}, \eta\tilde{U}).$$

By using the equation (2.14) with definition of conformality of Ψ, we deduce that

$$\begin{aligned} g_1(\nabla_{\tilde{X}}\tilde{Y}, \tilde{U}) = \quad & -\frac{1}{\lambda^2} g_2\left((\nabla\Psi_*)(\tilde{X}, L\tilde{Y}), \Psi_*\eta\tilde{U}\right) + \frac{1}{\lambda^2} g_2(\nabla^{\Psi}_{\tilde{X}}\Psi_* L\tilde{Y}, \Psi_*\eta\tilde{U}) \\ & + g_1(\nabla_{\tilde{X}}P\tilde{Y}, F\tilde{U}) + g_1(\mathcal{A}_{\tilde{X}}L\tilde{Y}, \xi\tilde{U}) \end{aligned}$$

Considering the (i) part of Lemma 2.1, above equation turns in to

$$\begin{aligned} g_1(\nabla_{\tilde{X}}\tilde{Y}, \tilde{U}) = \quad & \frac{1}{\lambda^2} g_2(\nabla^{\Psi}_{\tilde{X}}\Psi_* L\tilde{Y}, \Psi_*\eta\tilde{U}) + g_1(\nabla_{\tilde{X}}P\tilde{Y}, F\tilde{U}) + g_1(\mathcal{A}_{\tilde{X}}L\tilde{Y}, \xi\tilde{U}) \\ & - g_1(\operatorname{grad}\ln\lambda, \tilde{X}) g_1(L\tilde{Y}, \eta\tilde{U}) - g_1(\operatorname{grad}\ln\lambda, L\tilde{Y}) g_1(\tilde{X}, \eta\tilde{U}) \\ & + g_1(\operatorname{grad}\ln\lambda, \eta\tilde{U}) g_1(\tilde{X}, L\tilde{Y}). \end{aligned}$$

By direct calculation, finally we get

$$\begin{aligned} g_1(\tilde{X}, \tilde{Y})H = \quad & -P\mathcal{A}_{\tilde{X}}P\tilde{Y} - \xi\nabla_{\tilde{X}}P\tilde{Y} - \xi\mathcal{A}_{\tilde{X}}L\tilde{Y} - F\Psi_*(\nabla^{\Psi}_{\tilde{X}}\Psi_* L\tilde{Y}) + \tilde{X}(\ln\lambda)PL\tilde{Y} \\ & + L\tilde{Y}(\ln\lambda)P\tilde{X} - P(\operatorname{grad}\ln\lambda) g_1(\tilde{X}, L\tilde{Y}). \end{aligned}$$

From the above equation we conclude that $\bar{Q}_{1(\ker\Psi_*)^\perp}$ is totally umbilical if and only if equation (5.4) satisfied. This completes the proof of the theorem.

F

6 -Pluriharmonicity of Conformal Quasi Bi-slant Submersion

In this section, we extend the concept of F-pluriharmonicity to almost product Riemannian manifolds and definition of Hermitian manifold.

Definition 6.1. On a manifold M, a pair (J, g) consisting of a complex structure J on M and a Hermitian metric g in the tangent space TM, that is, a Riemannian metric g that is invariant under $J, g(JX, JY) = g(X, Y)$ for any vector fields X and Y on M. A Hermitian structure specifies in any tangent space T_pM the structure of a Hermitian vector space. A manifold with a Hermitian structure is called a Hermitian manifold.

Let Ψ be a CQBS Riemannian submersion from LPR manifold $(\bar{Q}_1, g_1, F)$ onto a Riemannian manifold $(\bar{Q}_2, g_2)$ with slant angles θ_1 and θ_2. Then CQBS Riemannian submersion is $\mathfrak{D}^{\mathfrak{T}} - F$-pluriharmonic, $\mathfrak{D}^{\theta_1} - F$-pluriharmonic, $\mathfrak{D}^{\theta_2} - F$-pluriharmonic, $\left(\mathfrak{D}^{\mathfrak{T}} - \mathfrak{D}^{\theta_1}\right) - F$ pluriharmonic, $\left(\mathfrak{D}^{\mathfrak{T}} - \mathfrak{D}^{\theta_2}\right) - F$ pluriharm (6.1)
$\ker\Psi_* - F$-pluriharmonic, $(\ker\Psi_*)^\perp - F$-pluriharmonic and $((\ker\Psi_*)^\perp - \ker\Psi_*) - F$-pluriharmonic if

$$(\nabla\Psi_*)(U, V) + (\nabla\Psi_*)(FU, FV) = 0,$$

for any $U, V \in \Gamma(\mathfrak{D}^{\mathfrak{T}})$, for any $U, V \in \Gamma\left(\mathfrak{D}^{\theta_1}\right)$, for any $U, V \in \Gamma\left(\mathfrak{D}^{\theta_2}\right)$, for any $U \in \Gamma(\mathfrak{D}^{\mathfrak{T}}), V \in \Gamma\left(\mathfrak{D}^{\theta_1}\right)$, for any $U \in \Gamma(\mathfrak{D}^{\mathfrak{T}}), V \in \Gamma\left(\mathfrak{D}^{\theta_2}\right)$, for any $U, V \in \Gamma(\ker\Psi_*)$, for any $U, V \in \Gamma(\ker\Psi_*)^\perp$ and for any $U \in \Gamma(\ker\Psi_*)^\perp, V \in \Gamma(\ker\Psi_*)$, respectively.

Theorem 6.1. Let Ψ be a CQBS submersion from LPR manifold $(\bar{Q}_1, g_1, F)$ onto a rm $(\bar{Q}_2, g_2)$ with slant angles θ_1 and θ_2. Suppose that Ψ is $\mathfrak{D}^{\theta_1} - F$ pluriharmonic. Then $\mathfrak{D}^{\theta_1}$ defines totally geodesic foliation on $\bar{Q}_1$ if and only

if

$$\begin{aligned}
&\Psi_*\big(\eta\mathcal{T}_{\xi\tilde{U}}\eta\xi\tilde{V} + L\mathcal{H}\nabla_{\xi\tilde{U}}\eta\xi\tilde{V}\big) - \Psi_*\big(\mathcal{A}_{\eta\tilde{U}}\xi\tilde{V} + \mathcal{H}\nabla_{\xi\tilde{U}}\eta\tilde{V}\big)\\
&= \cos^2\theta_1\Psi_*\big(L\mathcal{T}_{\xi\tilde{U}}\tilde{V} + \eta\mathcal{V}\nabla_{\xi\tilde{U}}\tilde{V}\big) + \nabla^{\Psi}_{\xi\tilde{U}}\Psi_*F\tilde{V}\\
&\ -\eta\tilde{U}(\ln\lambda)\Psi_*\eta\tilde{V} - \eta\tilde{V}(\ln\lambda)\Psi_*\eta\tilde{U} + g_1(\eta\tilde{U}, \eta\tilde{V})\Psi_*(\operatorname{grad}\ln\lambda)
\end{aligned}$$

for any $\tilde{U}, \tilde{V} \in \Gamma(\mathfrak{D}^{\theta_1})$.

Proof. For any $\tilde{U}, \tilde{V} \in \Gamma(\mathfrak{D}^{\theta_1})$ and since, Ψ is $D^{\theta_1} - F$-pluriharmonic, then by using equation (2.7) and (2.14), we have

$$\begin{aligned}
&0 = (\nabla\Psi_*)(\tilde{U}, \tilde{V}) + (\nabla\Psi_*)(F\tilde{U}, F\tilde{V})\\
&\Psi_*\big(\nabla_{\tilde{U}}\tilde{V}\big) = -\Psi_*\big(\nabla_{F\tilde{U}}F\tilde{V}\big) + \nabla^{\Psi}_{F\tilde{U}}\Psi_*(F\tilde{V})\\
&\ = -\Psi_*\big(\mathcal{A}_{\eta\tilde{U}}\xi\tilde{V} + \mathcal{V}\nabla_{\eta\tilde{U}}\xi\tilde{V} + \mathcal{T}_{\xi\tilde{U}}\eta\tilde{V} + \mathcal{H}\nabla_{\xi\tilde{U}}\eta\tilde{V}\big) - \Psi_*\big(F\nabla_{\xi\tilde{U}}F\xi\tilde{V}\big)\\
&\ +(\nabla\Psi_*)(\eta\tilde{U}, \eta\tilde{V}) - \nabla^{\Psi}_{\eta\tilde{U}}\Psi_*\eta\tilde{V} + \nabla^{\Psi}_{F\tilde{U}}\Psi_*F\tilde{V}.
\end{aligned}$$

By using equations (3.3), (3.6) with Lemma 2.1 and Lemma 3.2, the above equation finally takes the form

$$\begin{aligned}
\Psi_*(\nabla_{\tilde{U}}V) = \quad & -\cos^2\theta_1\Psi_*\big(P\mathcal{T}_{\xi\tilde{U}}\tilde{V} + L\mathcal{T}_{\xi\tilde{U}}\tilde{V} + \xi\mathcal{V}\nabla_{\xi\tilde{U}}\tilde{V} + \eta\mathcal{V}\nabla_{\xi\tilde{U}}\tilde{V}\big)\\
&+\Psi_*\big(\xi\mathcal{T}_{\xi\tilde{U}}\eta\xi\tilde{V} + \eta\mathcal{T}_{\xi\tilde{U}}\eta\xi\tilde{V} + P\mathcal{H}\nabla_{\xi\tilde{U}}\eta\xi\tilde{V} + L\mathcal{H}\nabla_{\xi\tilde{U}}\eta\xi\tilde{V}\big)\\
&-\Psi_*\big(\mathcal{A}_{\eta\tilde{U}}\xi\tilde{V} + \mathcal{V}\nabla_{\eta\tilde{U}}\xi\tilde{V} + \mathcal{T}_{\xi\tilde{U}}\eta\tilde{V} + \mathcal{H}\nabla_{\xi\tilde{U}}\eta\tilde{V}\big)\\
&+\eta\tilde{U}(\ln\lambda)\Psi_*\eta\tilde{V} + \eta\tilde{V}(\ln\lambda)\Psi_*\eta\tilde{U} - g_1(\eta\tilde{U}, \eta\tilde{V})\Psi_*(\operatorname{grad}\ln\lambda)\\
&-\nabla^{\Psi}_{\eta\tilde{U}}\Psi_*\eta\tilde{V} + \nabla^{\Psi}_{F\tilde{U}}\Psi_*F\tilde{V}
\end{aligned}$$

from which we get the desired result.

Theorem 6.2. Let Ψ be a CQBS submersion from LPR manifold $(\bar{Q}_1, g_1, F)$ onto a rm $(\bar{Q}_2, g_2)$ with slant angles θ_1 and θ_2. Suppose that Ψ is $\mathfrak{D}^{\theta_2} - F$ pluriharmonic. Then $\mathfrak{D}^{\theta_2}$ defines totally geodesic foliation on $\bar{Q}_1$ if and only if

$$\Psi_*\big(\eta\mathcal{T}_{\xi\tilde{Z}}\eta\xi\widetilde{W}+L\mathcal{H}\nabla_{\xi\tilde{Z}}\eta\xi\widetilde{W}\big)-\Psi_*\big(\mathcal{A}_{\eta\tilde{Z}}\xi\widetilde{W}+\mathcal{H}\nabla_{\xi\tilde{Z}}\eta\widetilde{W}\big)$$
$$=\cos^2\theta_2\Psi_*\big(L\mathcal{T}_{\xi\tilde{Z}}\widetilde{W}+\eta\widetilde{\mathcal{W}}\nabla_{\xi\tilde{Z}}\widetilde{W}\big)+\nabla^{\Psi}_{\xi\tilde{Z}}\Psi_*F\widetilde{W}$$
$$-\eta\tilde{Z}(\ln\lambda)\Psi_*\eta\widetilde{W}-\eta\widetilde{W}(\ln\lambda)\Psi_*\eta\tilde{Z}+g_1(\eta\tilde{Z},\eta\widetilde{W})\Psi_*(\operatorname{grad}\ln\lambda)$$

for any $\tilde{Z},\widetilde{W}\in\Gamma(\mathfrak{D}^{\theta_2})$.

Proof. The proof of the theorem is similar to the proof of Theorem 6.1.

Theorem 6.3. Let Ψ be a CQBS submersion from LPR manifold $(\bar{Q}_1,g_1,F)$ onto a rm $(\bar{Q}_2,g_2)$ with slant angles θ_1 and θ_2. Suppose that Ψ is $((ker\Psi_*)^\perp-\ker\Psi_*)-F$-pluriharmonic. Then the following assertion are equivalent.

(i) The horizontal distribution $(ker\Psi_*)^\perp$ defines totally geodesic foliation on $\bar{Q}_1$.

$$(\cos^2\theta_1+\cos^2\theta_2)\Psi_*\{L\mathcal{T}_{P\tilde{X}}\xi\mathfrak{A}\widetilde{U}+\eta\mathcal{V}\nabla_{P\tilde{X}}\xi\mathfrak{A}\widetilde{U}+L\mathcal{A}_{L\tilde{X}}\xi\mathfrak{A}\widetilde{U}+\eta\mathcal{V}\nabla_{L\tilde{X}}\xi\mathfrak{A}\widetilde{U}\}$$
$$=\Psi_*\{L\mathcal{T}_{P\tilde{X}}\xi\mathfrak{A}\widetilde{U}+\eta\mathcal{V}\nabla_{P\tilde{X}}\xi\mathfrak{A}\widetilde{U}+L\mathcal{A}_{L\tilde{X}}\xi\mathfrak{A}\widetilde{U}+\eta\mathcal{H}\nabla_{L\tilde{X}}\xi\mathfrak{A}\widetilde{U}\}$$
$$-\Psi_*\{\eta\mathcal{T}_{P\tilde{X}}\eta\xi\mathfrak{B}\widetilde{U}+L\mathcal{H}\nabla_{P\tilde{X}}\eta\xi\mathfrak{B}\widetilde{U}+\eta\mathcal{T}_{P\tilde{X}}\eta\xi\mathfrak{C}\widetilde{U}+L\mathcal{H}\nabla_{P\tilde{X}}\eta\xi\mathfrak{C}\widetilde{U}\}$$
$$+L\tilde{X}(\ln\lambda)\Psi_*\eta\xi\mathfrak{B}\widetilde{U}+\eta\xi\mathfrak{B}\widetilde{U}(\ln\lambda)\Psi_*L\tilde{X}-g_1(L\tilde{X},\eta\xi\mathfrak{B}\widetilde{U})\Psi_*(\operatorname{grad}\ln\lambda)$$
$$+L\tilde{X}(\ln\lambda)\Psi_*\eta\xi\mathfrak{C}\widetilde{U}+\eta\xi\mathfrak{C}\widetilde{U}(\ln\lambda)\Psi_*L\tilde{X}-g_1(L\tilde{X},\eta\xi\mathfrak{C}\widetilde{U})\Psi_*(\operatorname{grad}\ln\lambda)$$
$$-\Psi_*\{\eta\mathcal{A}_{L\tilde{X}}\eta\xi\mathfrak{B}\widetilde{U}+\eta\mathcal{A}_{L\tilde{X}}\eta\xi\mathfrak{C}\widetilde{U}-\mathcal{H}\nabla_{P\tilde{X}}\eta\widetilde{U}\}-\nabla^{\Psi}_{L\tilde{X}}\Psi_*\eta\xi\mathfrak{B}\widetilde{U}$$
$$+\Psi_*\big(\nabla_{\tilde{X}}\widetilde{U}\big)+\nabla^{\Psi}_{F\tilde{X}}\Psi_*\eta\widetilde{U}-\nabla^{\Psi}_{L\tilde{X}}\Psi_*\eta\xi\tilde{C}\widetilde{U},$$

(ii)

$$+L\tilde{X}(\ln\lambda)\Psi_*\eta\xi\mathfrak{B}\widetilde{U}+\eta\xi\mathfrak{B}\widetilde{U}(\ln\lambda)\Psi_*L\tilde{X}-g_1(L\tilde{X},\eta\xi\mathfrak{B}\widetilde{U})\Psi_*(\operatorname{grad}\ln\lambda)$$
$$+L\tilde{X}(\ln\lambda)\Psi_*\eta\xi\mathfrak{C}\widetilde{U}+\eta\xi\mathfrak{C}\widetilde{U}(\ln\lambda)\Psi_*L\tilde{X}-g_1(L\tilde{X},\eta\xi\mathfrak{C}\widetilde{U})\Psi_*(\operatorname{grad}\ln\lambda)$$
$$-\Psi_*\{\eta\mathcal{A}_{L\tilde{X}}\eta\xi\mathfrak{B}\widetilde{U}+\eta\mathcal{A}_{L\tilde{X}}\eta\xi\mathfrak{C}\widetilde{U}-\mathcal{H}\nabla_{P\tilde{X}}\eta\widetilde{U}\}-\nabla^{\Psi}_{L\tilde{X}}\Psi_*\eta\xi\mathfrak{B}\widetilde{U}$$
$$+\Psi_*\big(\nabla_{\tilde{X}}\widetilde{U}\big)+\nabla^{\Psi}_{F\tilde{X}}\Psi_*\eta\widetilde{U}-\nabla^{\Psi}_{L\tilde{X}}\Psi_*\eta\xi\tilde{C}\widetilde{U}$$

for any $\tilde{X}\in\Gamma(\ker\Psi_*)^\perp$ and $\widetilde{U}\in\Gamma(ker\Psi_*)$

Proof. For any $\tilde{X}\in\Gamma(ker\Psi_*)^\perp$ and $\widetilde{U}\in\Gamma(ker\Psi_*)$, since Ψ is $((ker\Psi_*)^\perp-ker\Psi_*)-F$-pluriharmonic, then by using (2.14), (3.3) and (3.6), we get

$$\Psi_*(\nabla_{L\tilde{X}}\eta\widetilde{U}) = -\Psi_*(\nabla_{P\tilde{X}}\xi\widetilde{U} + \nabla_{P\tilde{X}}\eta\widetilde{U} + \nabla_{L\tilde{X}}\xi\widetilde{U}) + \Psi_*(\nabla_{\tilde{X}}\widetilde{U}) + \nabla^{\Psi}_{F\tilde{X}}\Psi_*\eta\widetilde{U}.$$

Taking account the fact from (2.1) and (2.8), we have

$$\begin{aligned}\Psi_*(\nabla_{L\tilde{X}}\eta\widetilde{U}) = \quad & -\Psi_*(\mathcal{T}_{P\tilde{X}}\eta\widetilde{U} + \mathcal{H}\nabla_{P\tilde{X}}\eta\widetilde{U}) + \Psi_*(\nabla_{\tilde{X}}\widetilde{U}) + \nabla^{\Psi}_{F\tilde{X}}\Psi_*\eta\widetilde{U} \\ & -\Psi_*(F\nabla_{P\tilde{X}}F\xi\widetilde{U}) - \Psi_*(F\nabla_{L\tilde{X}}F\xi\widetilde{U})\end{aligned}$$

Now on using decomposition (3.2), Lemma 3.2, Lemma 3.3 with equations (3.3), we may yields

$$\begin{aligned}\Psi_*(\nabla_{L\tilde{X}}\eta\widetilde{U}) = \quad & \Psi_*\{F\nabla_{P\tilde{X}}\eta\xi\mathfrak{B}\widetilde{U} + F\nabla_{P\tilde{X}}\eta\xi\mathfrak{C}\widetilde{U} + F\nabla_{L\tilde{X}}\eta\xi\mathfrak{B}\widetilde{U} + F\nabla_{L\tilde{X}}\eta\xi\mathfrak{C}\widetilde{U}\} \\ & \Psi_*\{F\nabla_{P\tilde{X}}\xi\mathfrak{A}\widetilde{U} - \cos^2\theta_1 F\nabla_{P\tilde{X}}\xi\widetilde{U} - \cos^2\theta_2 F\nabla_{P\tilde{X}}\xi\widetilde{U}\} \\ & +\Psi_*\{F\nabla_{L\tilde{X}}\xi\mathfrak{A}\widetilde{U} - \cos^2\theta_1 F\nabla_{L\tilde{X}}\xi\widetilde{U} - \cos^2\theta_2 F\nabla_{L\tilde{X}}\xi\widetilde{U}\} \\ & -\Psi_*(\mathcal{H}\nabla_{P\tilde{X}}\eta\widetilde{U}) + \Psi_*(\nabla_{\tilde{X}}\widetilde{U}) + \nabla^{\Psi}_{F\tilde{X}}\Psi_*\eta\widetilde{U}\end{aligned}$$

From equations (2.7)-(2.10) and after simple calculation, we may write

$$\begin{aligned}\Psi_*(\nabla_{L\tilde{X}}\eta\widetilde{U}) = \quad & -(\cos^2\theta_1 + \cos^2\theta_2)\Psi_*\{L\mathcal{T}_{P\tilde{X}}\xi\mathfrak{A}\widetilde{U} + \eta\mathcal{V}\nabla_{P\tilde{X}}\xi\mathfrak{A}\widetilde{U} + L\mathcal{A}_{L\tilde{X}}\xi\mathfrak{A}\widetilde{U} \\ & +\eta\mathcal{V}\nabla_{L\tilde{X}}\xi\mathfrak{A}\widetilde{U}\} - \Psi_*\{\eta\mathcal{A}_{L\tilde{X}}\eta\xi\mathfrak{B}\widetilde{U} + \eta\mathcal{A}_{L\tilde{X}}\eta\xi\mathfrak{C}\widetilde{U} - \mathcal{H}\nabla_{P\tilde{X}}\eta\widetilde{U}\} \\ & +\Psi_*\{L\mathcal{T}_{P\tilde{X}}\xi\mathfrak{A}\widetilde{U} + \eta\mathcal{V}\nabla_{P\tilde{X}}\xi\mathfrak{A}\widetilde{U} + L\mathcal{A}_{L\tilde{X}}\xi\mathfrak{A}\widetilde{U} + \eta\mathcal{H}\nabla_{L\tilde{X}}\xi\mathfrak{A}\widetilde{U}\} \\ & -\Psi_*\{\eta\mathcal{T}_{P\tilde{X}}\eta\xi\mathfrak{B}\widetilde{U} + L\mathcal{H}\nabla_{P\tilde{X}}\eta\xi\mathfrak{B}\widetilde{U} + \eta\mathcal{T}_{P\tilde{X}}\eta\xi\mathfrak{C}\widetilde{U} + L\mathcal{H}\nabla_{P\tilde{X}}\eta\xi\mathfrak{C}\widetilde{U}\} \\ & -\Psi_*(L\mathcal{H}\nabla_{L\tilde{X}}\eta\xi\mathfrak{B}\widetilde{U} + L\mathcal{H}\nabla_{L\tilde{X}}\eta\xi\mathfrak{B}\widetilde{U}) + \Psi_*(\nabla_{\tilde{X}}\widetilde{U}) + \nabla^{\Psi}_{F\tilde{X}}\Psi_*\eta\widetilde{U}\end{aligned}$$

Since Ψ is conformal Riemannian submersion, the by using equations (2.14) and from Lemma 2.1, we finally have

$$\begin{aligned}&\Psi_*(\nabla_{L\tilde{X}}\eta\widetilde{U}) = -(\cos^2\theta_1 + \cos^2\theta_2)\Psi_*\{L\mathcal{T}_{P\tilde{X}}\xi\mathfrak{A}\tilde{U} + \eta\mathcal{V}\nabla_{P\tilde{X}}\xi\mathfrak{A}\widetilde{U} + L\mathcal{A}_{L\tilde{X}}\xi\mathfrak{A}\widetilde{U} \\ &+\eta\mathcal{V}\nabla_{L\tilde{X}}\xi\mathfrak{A}\widetilde{U}\} - \Psi_*\{\eta\mathcal{A}_{L\tilde{X}}\eta\xi\mathfrak{B}\widetilde{U} + \eta\mathcal{A}_{L\tilde{X}}\eta\xi\mathfrak{C}\widetilde{U} - \mathcal{H}\nabla_{P\tilde{X}}\eta\widetilde{U}\} \\ &+\Psi_*\{L\mathcal{T}_{P\tilde{X}}\xi\mathfrak{A}\tilde{U} + \eta\mathcal{V}\nabla_{P\tilde{X}}\xi\mathfrak{A}\tilde{U} + L\mathcal{A}_{L\tilde{X}}\xi\mathfrak{A}\tilde{U} + \eta\mathcal{H}\nabla_{L\tilde{X}}\xi\mathfrak{A}\widetilde{U}\} \\ &-\Psi_*\{\eta\mathcal{T}_{P\tilde{X}}\eta\xi\mathfrak{B}\widetilde{U} + L\mathcal{H}\nabla_{P\tilde{X}}\eta\xi\mathfrak{B}\widetilde{U} + \eta\mathcal{T}_{P\tilde{X}}\eta\xi\mathfrak{C}\widetilde{U} + L\mathcal{H}\nabla_{P\tilde{X}}\eta\xi\mathfrak{C}\widetilde{U}\} \\ &+L\tilde{X}(\ln\lambda)\Psi_*\eta\xi\mathfrak{B}\widetilde{U} + \eta\xi\mathfrak{B}\widetilde{U}(\ln\lambda)\Psi_*L\tilde{X} - g_1(L\tilde{X}, \eta\xi\mathfrak{B}\widetilde{U})\Psi_*(\text{grad}\ln\lambda) \\ &+L\tilde{X}(\ln\lambda)\Psi_*\eta\xi\tilde{C}\tilde{U} + \eta\xi\tilde{C}\tilde{U}(\ln\lambda)\Psi_*L\tilde{X} - g_1(L\tilde{X}, \eta\xi\tilde{C}\tilde{U})\Psi_*(\text{grad}\ln\lambda) \\ &+\Psi_*(\nabla_{\tilde{X}}\widetilde{U}) + \nabla^{\Psi}_{F\tilde{X}}\Psi_*\eta\widetilde{U} - \nabla^{\Psi}_{L\tilde{X}}\Psi_*\eta\xi\mathfrak{B}\widetilde{U} - \nabla^{\Psi}_{L\tilde{X}}\Psi_*\eta\xi\mathfrak{C}\widetilde{U},\end{aligned}$$

which completes the proof of theorem.

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