A Note on Generalization of Totally Projective Modules

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Abstract: In this paper we extend the notion of totally projective QTAG- modules as almost totally projective QTAG-modules and prove that for an h- reduced QTAG-module M with a submodule N such that M/N is countably generated; if N is almost totally projective module then M is also almost totally projective.

Keywords: QTAG Modules, h-reduced QTAG-module, totally projective module, nice submodule, 2020 Mathematics Subject Classification: 20K10.

عنوان البحث: ملاحظة حول تعميم وحدات الإسقاط الكلي

الملخص: في هذا البحث ، نوسع مفهوم وحدات QTAG الإسقاطية تمامًا كوحدات QTAG إسقاطيه بالكامل تقريبًا ونثبت أنه بالنسبة للوحدة M المختصرة من QTAG مع وحدة فرعية N بحيث يتم إنشاء M / N بشكل عددي ؛ إذا كانت N عبارة عن وحدة إسقاطيه بالكامل تقريبًا

1 Introduction

Many concepts for groups like purity, projectivity, injectivity, height etc. have been generalized for modules. To obtain results of groups which are not true for modules either conditions have been applied on modules or upon the underlying rings. We imposed the condition on modules that every finitely generated submodule of any homomorphic image of the module is a direct sum of uniserial modules while the rings are associative with unity. After these conditions, many elegant results of groups can be proved for QTAG-modules which are not true in general. This paper take the motivation for generalization from the results in the paper [1].

The study of QTAG-modules was initiated by Singh [8]. Khan [5], Mehdi [7] etc. worked a lot on these modules etc. They studied different notions and structures of QTAG-modules, developed the theory of these modules by introducing several notions and investigated some interesting properties and characterized them. Yet there is much to explore.

A module M over an associative ring R with unity is a QTAG-module if every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules [9]. All the rings R considered here are associative with unity and modules M are until QTAG-modules. An element x ∈ M is uniform, if xR is a non-zero uniform (hence uniserial) module and for any R-module M with a unique composition series, d(M) denotes its composition length. For a uniform element x ∈ M, e(x) = d(xR) and $H_M(x) = \sup\left\{d\left(\frac{yR}{xR}\right) \mid y \in M, x \in yR \text{ and } y \text{ uniform}\right\}$ are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k. M is h-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$[5] and it is h-reduced if it does not contain any h-divisible submodule. In other words it is free from the elements of infinite height. A submodule N of a QTAG-module M is a nice submodule if every nonzero coset $a + N$ is proper with respect to N i.e. for every nonzero $a + N$ there is an element $b \in N$ such that $H_M(a + b) = H_{M/N}(a + N)$. A family $\mathcal{N}$ of submodules of M is called a nice system in M if

(i) $0 \in \mathcal{N}$; (ii) If $\{N_i\}_{i \in I}$ is any subset of $\mathcal{N}$, then $\Sigma_I N_i \in \mathcal{N}$; (iii) Given any $N \in \mathcal{N}$ and any countable subset X of M, there exists $K \in \mathcal{N}$ containing $N \cup X$, such that K/N is countably generated [7].

Every submodule in a nice system is nice submodule. A h-reduced $QTAG$ module M is called totally projective if it has a nice system. Notations and terminology are follows from [3].

2 Main Results

Totally projective modules were defined by Mehdi [7]. Hasan [4], etc. worked a lot on these modules. Here we start with the generalization of these modules as almost totally projective QTAG-modules.

Definition 1 A h-reduced QTAG-module M is almost totally projective if it has a collection A of nice submodules such that

(i) $\{0\} \in \mathcal{A}$;
(ii) $\mathcal{A}$ is closed with respect to unions of ascending chains and
(iii) if N is a countably generated submodule of M then there exists $K \in \mathcal{A}$ such that $N \subseteq K$ and K is also countably generated.

First we will establish some elementary results related to almost totally projective QTAG-modules. We start with the following:

Proposition 1 Suppose K is an isotype submodule of a QTAG-module M. Then K is almost totally projective provided that K is separable in M.

Proof Let K be an almost totally projective QTAG-module and suppose to the contrary that it is not separable in M. Then, there exists $m \in M$ such that, for each countably generated submodules T of K, we can find an element $k^* \in K$ such that $H(m + k^*) > H(m + t)$ | for every $t \in T$. Therefore we can find an ascending chain

$$0 = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n \subseteq \cdots$$

of countably generated submodules K_n of K such that $K_n \in \mathcal{A}$ for each n and such that the following condition holds.

(*) For every finite n, there exists $k_{n+1} \in K_{n+1}$ such that $H(m + k_{n+1}) > H(m + k)$ for all $k \in K_n$.

Now set $K_\omega = \bigcup_{n<\omega_0} K_n$ and observe that K_ω is a countably generated submodule of K belonging to $\mathcal{A}$. Since K_ω is countably generated, there exists $k^* \in K$ such that $H(m+t) < H(m+k^*)$ for every $t \in K_\omega$. Since K_ω is nice in K, there exists $k' \in K_\omega$ such that $H_K(k^* - t) \leqslant H_K(k^* - k')$ for all $t \in K_\omega$. Moreover, since $k' \in K_n$ for some n, there exists by condition ($\star$) an element $k'' \in K_{n+1}$ such that $H(m + k'') < H(m + k^*)$. So, we now have $H_K(k^* - k'') \leq H_K(k^* - k')$. Since K is isotype in M, we have $H(m + k') < H\big((m + k^*) - (m + k'')\big) = H(k^* - k'') = H_K(k^* - k'') \leq H_K(k^* - k') = H(k^* - k') = H(m + k')$ which leads to a contradiction and proving our desired result.

The next two propositions as analogs of the corresponding well-known results for totally projective modules.

Proposition 2 If $H_\sigma(M)$ and $M/H_\sigma(M)$ are almost totally projective, for any ordinal σ then M is also almost totally projective.

Proof We know that a submodule K is a nice submodule of M if and only if $pH_\sigma(K)$ is a nice submodule of $H_\sigma(K)$ and $K + H_\sigma(M)/H_\sigma(M)$ is a nice submodule of $M/H_\sigma(M)$. Hence, the properties that satisfy the three conditions for a family of nice submodules to be almost totally projective for both $H_\sigma(M)$ and $M/H_\sigma(M)$ also satisfy the same conditions for the module M.

Proposition 3: The arbitrary direct sums of almost totally projective QTAGmodules are almost totally projective.

Proof: Suppose $M = \oplus_{i\in I} M_i$. As it is well known, if $K = \oplus_{i\in I} K_i$ with $K_i \subseteq M_i$ for all $i \in I$, then K is a nice submodule of M if and only if K_i is a nice subgroup of M_i

for each $i \in I$. Hence, the three properties of almost totally projectives modules satisfying by M_i ensures that M will certainly satifies the same and hence the result follows.

We proceed by proving our main theorems. Before doing that, we need the following useful technicality on niceness, which can be of general interest as well.

Lemma 1: Let M be a $QTAG$-module with a nice submodule N. If K is a $QTAG$-module such that $K \cap M \subseteq N$, then $N + K$ is nice in $M + K$.

Proof: We apply transfinite induction on the ordinals, to prove that $H_\alpha(M + K) \subseteq H_\alpha(M) + K + N$ for every ordinal α.

If α is not a limit ordinal then by using induction we express

$$\begin{aligned} H_\alpha(M + K) &= H_1(H_{\alpha-1}(M + K)) = H_1(H_{\alpha-1}(M) + K + N) \\ &\subseteq H_1(H_{\alpha-1}(M)) + K + N = H_\alpha(M) + K + N. \end{aligned}$$

If α is a limit ordinal then again by transfinite induction we have

$$H_\alpha(M + K) = \bigcap_{\beta<\alpha} H_\beta(M + K) \subseteq \bigcap_{\beta<\alpha} \left(H_\beta(M) + K + N\right).$$

Let $u \in \cap_{\beta<\alpha}\left(H_\beta(M) + K + N\right)$. Then $u = x_\rho + y_1 + z_1 = x_\sigma + y_2 + z_2 = \cdots$ where $x_\rho \in H_\rho(M), y_1 \in K, z_1 \in N; x_\sigma \in H_\sigma(M), y_2 \in K, z_2 \in N, \rho < \sigma < \alpha$. Since $M \cap K \subseteq N, y_2 \in y_1 + N, x_\rho + z_1 \in \cap_{\beta<\alpha}\left(N + H_\beta(M)\right) = N + H_\alpha(M)$ because N is nice in M. Therefore $u \in H_\alpha(M) + K + N$, thus $\cap_{\beta<\alpha}\left(H_\beta(M) + K + N\right) \subseteq H_\alpha(M) + K + N$
Now we may conclude that, for an ordinal γ

$$\begin{aligned} \bigcap_{\gamma<\delta} \left(N + K + H_\gamma(M + K)\right) &= \bigcap_{\gamma<\delta} \left(N + K + H_\gamma(M)\right) \\ &= N + K + H_\delta(M) \\ &= N + K + H_\delta(M + K). \end{aligned}$$

Therefore $N + K$ is nice in $M + K$.

Now we are able to prove the following:

Theorem 1: Let M be a h-reduced $QTAG$-module with a submodule N such that M/N is countably generated. If N is almost totally projective module then M is also almost totally projective.

Proof: We may express $M = N + K$ where K is countably generated. Since N is almost totally projective it has a collection $\mathcal{B}$ of nice submodules satisfying the three conditions of Definition 1. Thus there exists a countably generated submodule T of $N \in \mathcal{B}$ such that $N \cap K \subseteq T$. Now $K + T$ is again countably generated and we may write $(K + T) \cap N = T + K \cap N = T$. Also $\frac{M}{K+T} = \frac{N+K}{K+T} \simeq \frac{N}{N\cap(K+T)} = \frac{N}{T}$. Since T is countably generated, N is almost totally projective where $\mathcal{F} = \{P/T \mid T \subseteq P \in \mathcal{B}\}$ is the collection of nice submodules of N/T. By the same argument $\frac{M}{K+T}$ has a collection $\mathcal{A}$ of nice submodules $\frac{Q}{K+T}$ such that the three conditions are satisfied and it is almost totally projective. Now we put $\mathcal{A}' = \{0\} \cup \left\{Q \subseteq M \mid \frac{Q}{K+T} \in \mathcal{A}\right\}$.

Now $\frac{Q}{K+T}$ is nice in $\frac{M}{K+T}$ and by Lemma 1, $K + T$ is nice in $K + N = M$ then by [6], Q is a nice submodule of M. Now $\{0\} \in \mathcal{A}'$ by definition. Consider $\{Q_i\}_{i\in I}$ the ascending chain of members of $\mathcal{A}'$. Since $\left\{\frac{Q_i}{K+T}\right\}$ is also an ascending chain for all indices i, we find that $\cup_{i\in I}\left(\frac{Q_i}{K+T}\right) = \frac{\cup_{i\in I} Q_i}{K+T} \in \mathcal{A}$, hence $\cup Q_i \in \mathcal{A}'$. Now suppose S is a countably generated submodule of M. Thus $\frac{S+K+T}{K+T} \simeq \frac{S}{S\cap(K+T)}$ is a countably generated submodule of $\frac{M}{K+T}$ and there is $\frac{Q}{K+T} \in \mathcal{A}$ such that $\frac{S+K+T}{K+T} \subseteq \frac{Q}{K+T}$ where $\frac{Q}{K+T}$ is countably generated. Therefore $S + K + T \subseteq Q$, hence $S \subseteq Q$ where $Q \in \mathcal{A}'$ and Q is countably generated, so is $K + T$ and all the conditions are satisfied.

Theorem 2: Let M be a h-reduced $QTAG$-module with a nice countably generated submodule N such that M/N is almost totally projective. Then M is almost totally projective.

Proof: Let M/N be an almost totally projective module. Now M/N has a collection $\mathcal{B}$ of nice submodules satisfying the three conditions. Consider $\mathcal{A} = \left\{K \subseteq M \mid \frac{K}{N} \in \mathcal{B}\right\} \cup \{0\}$.

If $0 \neq K \in \mathcal{A}$ then $K/N \in \mathcal{B}$ and hence K/N is nice inM/N. Since N is nice in M by [6], K is nice in M. Therefore $\mathcal{A}$ is the collection of nice submodules of M such that$\{0\} \in \mathcal{A}$.

Let $\{T_i\}_{i\in I}$ be an ascending chain of nonzero modules, belonging to $\mathcal{A}$. Now $\left\{\frac{T_i}{N}\right\}$ is also an ascending chain of modules belonging to $\mathcal{B}$. Now $\cup_{i\in I}\left(\frac{T_i}{N}\right) = \frac{\cup_{i\in I} T_i}{N} \in \mathcal{B}$. Therefore $\cup T_i \in \mathcal{A}$ as $\cup T_i \subseteq M$ and the second condition is satisfied. Let Q be an arbitrary countably generated submodule of M. Now $\frac{Q+N}{N} \simeq \frac{Q}{Q\cap N}$ is a countably generated submodule of M/N and there is $P/N \in \mathcal{B}$ such that $\frac{Q+N}{N} \subseteq P/N$ and P/N is countably generated. Therefore $P \in \mathcal{A}, Q + N \subseteq P \subseteq \mathcal{A}$, hence $Q \subseteq P$ and P is countably generated, so is N. When $Q \subseteq N$, we may take $P = N$. Since $N \in \mathcal{A}$ and N is countably generated.

Following is the immediate consequence of the above results.

Corollary 1 Let N be a submodule of a h-reduced $QTAG$-module M with length $< \sigma$. If N is separable and the direct sum of countably generated modules such that M/N is countably generated or N is countably generated and nice in M such that M/N is also separable and a direct sum of countably generated modules then M is also separable and the direct sum of countably generated modules.

We end this short communication by posting an open problem as follows:

Open Problem: Under what suitable restrictions on a submodule N, if we suppose M/N be an almost totally projective QTAG-module, which leads to a necessary and sufficient condition for M to be almost totally projective QTAG-module.

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