Fundamentals to Detect Tensor Product Bézier Patches in Euclidean 3-Space

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Abstract

The aim of this paper is to deliver the fundamentals to detect Bézier patches of scanned objects based on their normal congruence. In five-dimensional real projective space (P^5), we introduce a new approach for tensor product (TP) Bézier patch representation. For this reason, we use Plücker coordinates which are a way to assign six homogeneous coordinates to each line in three-dimensional projective space (P^3). Derivatives, normal vectors of Bézier patches and some of geometric properties of these patches are discussed. Further, the special case, biquadratic Bézier patch is introduced. The Plücker coordinates of the normal congruence of the patch are functions of order 14 in general, because of that high degree, it seems not to be of practical use to calculate the focal points of the normal vectors of the patch in general. We try these calculations for the biquadratic patches ($m=n=2$). Finally, we present a computational example to compute the two focal points of a normal of this patch.

Keywords

TP-surfaces, Curvature lines, Normal congruence, Focal surfaces of congruence, Plücker coordinates.

أساسيات تحديد قطع سطوح بيزيه في الفراغ الإقليدي ثلاثي البعد

الهدف من هذا البحث هو تقديم الأساسيات لتحديد قطع سطوح بيزيه من المجسمات الممسوحة ضوئيًا بناءً على أعمدتها المتوافقة. في الفضاء الإسقاطي ثلاثي الأبعاد (P5)، نقدم طريقة جديدة لتمثيل Tensor (TP) Bézier patch. لهذا الغرض، نستخدم إحداثيات *Plücker* التي تعد وسيلة لتعيين ست إحداثيات متجانسة لكل خط مستقيم في الفضاء الإسقاطي ثلاثي الأبعاد (P3). وتم مناقشة المشتقات والمتجهات العمودية على هذا النوع من السطوح وبعض الخصائص الهندسية لهذه القطع السطحية. علاوة على ذلك، تم تقديم الحالة الخاصة biquadratic Bézier patch وبسبب الدرجة العالية للدوال المستخدمة، ستبدو الحسابات بدون فائدة بشكل عام لذلك نحن نحاول عمل هذه الحسابات للقطع السطحية الخاصة ذات البعد الثنائي للدوال المستخدمة biquadratic (m=n=2). أخيرًا، قمنا بتقديم مثالاً حسابيًا لحساب نقطتين بؤريتين للأعمدة عند النقاط الحدية للسطح ورسمنا هذه النقاط والسطوح باستخدام أحد حزم البرامج الرياضية الجاهزة.

1. Introduction

Space curves and their frames is important in differential geometry, Mechanics and Physics. They have many applications in Computer Aided Design (CAD), Computer Aided Geometric Design (CAGD) (for more details, see [1-6]).

In reverse engineering the detection of special classes of CAGD generated surfaces is often based on the line congruence of normals and the GAUSS-image of such a surface. For surfaces with a "kinematic generation", as example helical surfaces or surfaces of revolution, the congruence of normals belong to a linear complex and this fact allows the detection via a line geometric treatment [7, 10]. Patch detection in case of general spline surfaces is not yet solved successfully. While the patch generation is at most invariant with respect to affine transformation, the normal congruence of the patch is a Euclidean concept. Therefore, we can expect that connections between the patch and its congruence turn to be rather complicated already for (algebraic) Bézier surfaces.

The analysis of the patch's normal congruence is only the first step of the far more difficult investigation of a reflection or refraction congruence with respect to a given patch. This problem belongs to geometric optics which would have explicit industrial applications but is not the topic of this paper. We restrict ourselves to an analysis of the normal congruence of a TP- Bézier patch [4, 7, 10].

A Bézier curve was named after Pierre Bézier, an engineer and mathematician who developed this method of computer drawing in late 1960s while working for the car manufacturer Règie Renault [3].

Derivatives and normal vectors of these curves are important issues in geometric modeling and computer graphics [10]. The first derivative of a degree *n* polynomial Bézier curve can be expressed as a degree *n-1* polynomial Bézier curve. For a tensor-product Bézier patch of degree *m* in *r*-parameter and *n* in s-parameter, the partial derivatives with respect to each parameter *r* or *s* are also tensor-product Bézier patches of degree *m-1* in *r* and *n* in *s* or *m* in *r* and *n-1* in *s*. The normal direction can be obtained as the cross product of the partial derivative respect to *r* and the partial derivative respect to *s*, its degree is $(2m-1)\times(2n-1)$.

2. Differential geometry of TP- Bézier patches

A tensor product Bézier patch Φ of degree (*m, n*) and P_{ij} control points can be defined as follows:

$$X^{m,n}(u,v) = \sum_{i=0}^{m}\sum_{j=0}^{n} B_i^m(u)\, B_j^n(v)\; P_{i\,j} \;\; ; \; u,v \in [0,1]\times[0,1], \tag{1}$$

whereby $B_i^m(u)$ *and* $B_j^n(v)$ denote the Bernstein polynomials of degree *m* and *n* in *u* and *v* parameters respectively. We will assume that the set of control points is chosen such that

$$X_u \times X_v \neq 0 \quad \forall \;\; (u,v) \in [0,1]\times[0,1]. \tag{2}$$

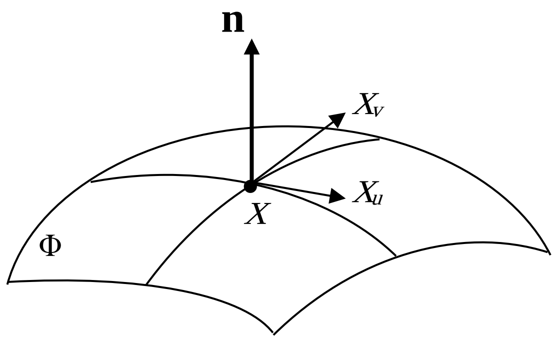

Figure 1. A Parametric surface patch.

We want to calculate the first fundamental form:

$$ds^2 = E\, du^2 + 2F\, du\, dv + G\, dv^2 , \tag{3}$$

of that patch, that means we have to calculate

$$E = X_u\, X_u \;,\;\; F = X_u\, X_v \;,\;\;\; G = X_v\, X_v \;, \tag{4}$$

where

$$X_u = \frac{\partial}{\partial u} X^{m,n}(u,v) = m \sum_{j=0}^{n}\sum_{i=0}^{m-1} B_i^{m-1}(u) B_j^n(v)\, \Delta^{1,0}\, P_{i,j} \quad ,$$

$$X_v = \frac{\partial}{\partial v} X^{m,n}(u,v) = n \sum_{i=0}^{m}\sum_{j=0}^{n-1} B_j^{n-1}(v) B_i^m(u)\; \Delta^{0,1}\, P_{i,j}. \tag{5}$$

The partials *Xu* and *Xv* at a point *X* span the tangent plane to the patch at *X*.
Let *Y* be any point on this plane. Then

$$\det[Y - X, X_u, X_v] = 0, \tag{6}$$

is the implicit equation of the tangent plane. The parametric equation is

$$Y(u,v) = X + \lambda\, X_u + \mu\, X_v \;\; ; \; \lambda, \mu \in \Re. \tag{7}$$

The normalized normal

$$\boldsymbol{n} = \frac{X_u \times X_v}{\left\| X_u \times X_v \right\|} \quad , \tag{8}$$

together with the unnormalized vectors *Xu* and *Xv* form a local coordinate system, a frame, at the point $X \in \Phi$ (see Fig. 2).

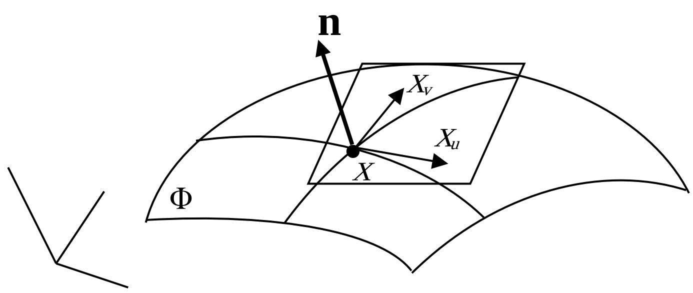

Figure 2. The local frame and the tangent plane.

From curve theory [7, 8], we know that its curvature κ is defined by $t' = \kappa\, \boldsymbol{m}$.

We now calculate the second fundamental form of Φ,

$$\kappa \cos\phi\, ds^2 = L\, du^2 + 2M\, du\, dv + N\, dv^2, \tag{9}$$

whereby ϕ is the angle between the main normal *m* of the curve $c \subset \Phi$ and the surface normal *n* and κ its curvature at the point X under consideration, as illustrated in Fig. 3.

Here, *L*, *M* and *N* are defined as follows:

$$\begin{aligned} L &= \boldsymbol{n} X_{uu} , \\ M &= \boldsymbol{n} X_{uv} , \\ N &= \boldsymbol{n} X_{vv} . \end{aligned} \tag{10}$$

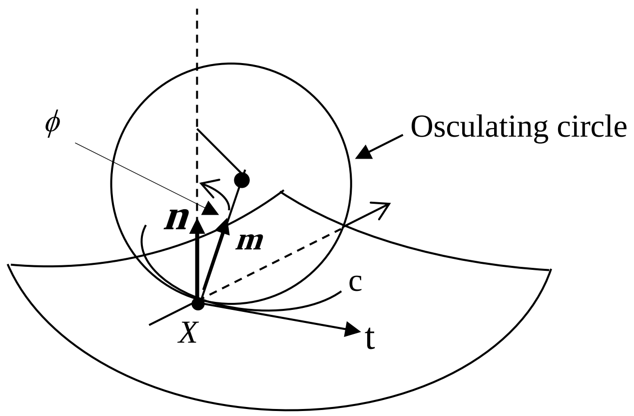

Figure 3. Osculating circle.

The 2nd derivatives are the expressions, since

$$\frac{\partial^r}{\partial u^r} X^{m,n}(u,v) = \frac{m!}{(m-r)!}\sum_{j=0}^{n}\sum_{i=0}^{m-r} B_i^{m-r}(u)\, B_j^n(v)\, \Delta^{r,0} P_{i,j},$$

$$\frac{\partial^s}{\partial v^s} X^{m,n}(u,v) = \frac{n!}{(n-s)!}\sum_{i=0}^{m}\sum_{j=0}^{n-s} B_j^{n-s}(v)\; B_i^m(u)\, \Delta^{0,s} P_{i,j},$$

$$\frac{\partial^{r+s}}{\partial u^r \partial v^s} X^{m,n}(u,v) = \frac{m!\,n!}{(m-r)!(n-s)!}\sum_{i=0}^{m-r}\sum_{j=0}^{n-s} B_i^{m-r}(u)\, B_j^{n-s}(v)\, \Delta^{r,s} P_{i,j}, \tag{11}$$

where

$$\Delta^{r,0} P_{i,j} = \Delta^{r-1,0} P_{i+1,j} - \Delta^{r-1,0} P_{i,j},$$

$$\Delta^{0,s} P_{i,j} = \Delta^{0,s-1} P_{i,j+1} - \Delta^{0,s-1} P_{i,j}. \tag{12}$$

Now, we get

$$X_{uu} = \frac{\partial^2}{\partial u^2} X^{m,n}(u,v) = \frac{m!}{(m-2)!}\sum_{j=0}^{n}\sum_{i=0}^{m-2} B_i^{m-2}(u)\, B_j^n(v)\, \Delta^{2,0} P_{i,j},$$

$$X_{uv} = \frac{\partial^2}{\partial u\ \partial v} X^{m,n}(u,v) = \frac{m!\,n!}{(m-1)!(n-1)!}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1} B_i^{m-1}(u)\, B_j^{n-1}(v)\, \Delta^{1,1} P_{i,j},$$

$$X_{vv} = \frac{\partial^2}{\partial v^2} X^{m,n}(u,v) = \frac{n!}{(n-2)!}\sum_{i=0}^{m}\sum_{j=0}^{n-2} B_j^{n-2}(v)\; B_i^m(u)\, \Delta^{0,2} P_{i,j}, \tag{13}$$

whereby

$$\Delta^{1,1} P_{i,j} = (P_{i+1,j+1} - P_{i+1,j}) - (P_{i,j+1} - P_{i,j}),$$

$$\Delta^{2,0} P_{i,j} = \Delta^{1,0} P_{i+1,j} - \Delta^{1,0} P_{i,j}\ ,\ \ \Delta^{1,0} P_{i,j} = P_{i+1,j} - P_{i,j}\ ,$$

$$\Delta^{0,2} P_{i,j} = \Delta^{0,1} P_{i,j+1} - \Delta^{0,1} P_{i,j}\ ,\ \ \Delta^{0,1} P_{i,j} = P_{i,j+1} - P_{i,j}. \tag{14}$$

So, for $\phi = 0$, the osculating plane of the curve is perpendicular to the patch tangent plane at the point *X*. The curvature of such a curve is called the normal curvature of the surface patch at *X* and given by

$$\kappa_0 = \frac{2^{nd}\ fundamentd\ form}{1^{st}\ fundamentd\ form}. \tag{15}$$

Now, we can calculate principal curvature lines through a fixed point $X \in \Phi$, which belong to directions du/dv, where κ_0 is extremal. Setting $\lambda = dv/du$, we can write Eq. (15) as

$$\kappa_0(\lambda) = \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2}, \tag{16}$$

such that $\kappa = \kappa_0(\lambda)$ is a rational quadratic function in λ. The extreme values κ_1 *and* κ_2 of $\kappa(\lambda)$ occur at the roots λ_1 *and* λ_2 of

$$\det\begin{bmatrix} \lambda^2 & -\lambda & 1 \\ E & F & G \\ L & M & N \end{bmatrix} = 0 \tag{17}$$

$$\Rightarrow \lambda^2 (FN - MG) + \lambda(EN - LG) + (EM - LF) = 0$$

$$\Rightarrow \lambda_1, \lambda_2 = \frac{1}{2(FN - MG)}\left\{(LG - EN) \pm \sqrt{(EN - LG)^2 - 4(FN - MG)(EM - LF)}\right\}. \tag{18}$$

Also, we can calculate the Gaussian and mean curvatures of the patch as follows:

$$\kappa_1 \kappa_2 = \frac{LN - M^2}{EG - F^2},$$

$$\kappa_1 + \kappa_2 = \frac{NE - 2MF + LG}{EG - F^2}. \tag{19}$$

Where, the term $K = \kappa_1 \kappa_2$ is called Gaussian curvature, while $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ is called mean curvature. Note that, both κ_1 *and* κ_2 change sign if the normal *n* is reversed, but *K* is not affected by such a reversal.

3. Motivation and results

In case of bicubic patch (*m*=*n*=*3*), we receive for $X_u, X_v, X_{uu}, X_{uv}, X_{vv}$, where

$$X_u = 3\sum_{j=0}^{3}\sum_{i=0}^{2} B_i^2(u) B_j^3(v)\Delta^{1,0} P_{i,j} = 3\sum_{j=0}^{3}\sum_{i=0}^{2} B_i^2(u) B_j^3(v)(P_{i+1,j} - P_{i,j}),$$

$$X_v = 3\sum_{i=0}^{3}\sum_{j=0}^{2} B_j^2(v) B_i^3(u)\, \Delta^{0,1} P_{i,j} = 3\sum_{i=0}^{3}\sum_{j=0}^{2} B_j^2(v) B_i^3(u)(P_{i,j+1} - P_{i,j}),$$

$$X_{uu} = 6\sum_{j=0}^{3}\sum_{i=0}^{1} B_i^1(u) B_j^3(v)\Delta^{2,0} P_{i,j} = 6\sum_{j=0}^{3}\sum_{i=0}^{1} B_i^1(u) B_j^2(v)\,(P_{2,j} - 2P_{1,j} + P_{0,j}),$$

$$X_{vv} = 6\sum_{i=0}^{3}\sum_{j=0}^{1} B_j^1(v)\,B_i^3(u)\,\Delta^{0,2}\,P_{i,j} = 6\sum_{i=0}^{3}\sum_{j=0}^{1} B_j^1(v)B_i^3(u)\,(P_{i,2} - 2P_{i,1} + P_{i,0})\,,$$

$$X_{uv} = 9\sum_{i=0}^{2}\sum_{j=0}^{2} B_i^2(u)\,B_j^2(v)\,\Delta^{1,1}\,P_{i,j} = 9\sum_{i=0}^{2}\sum_{j=0}^{2} B_i^2(u)\,B_j^2(v)[(P_{i+1,j+1} - P_{i+1,j}) - (P_{i,j+1} - P_{i,j})]\,.$$

That means: the Plücker coordinates $\left(X_u \times X_v\,; \mathrm{X} \times (X_u \times X_v)\right)\Re$ of the normal congruence of the patch are functions of order 14 because of that high degree, it seems not to be of practical use to calculate the focal points of the normal vectors of the patch in general. We apply these calculations to a biquadratic patch Φ:

Specializing $m = n = 2$ we receive

$$X_u = 2\sum_{j=0}^{2}\sum_{i=0}^{1} B_i^1(u)B_j^2(v)\,\Delta^{1,0}\,P_{i,j} = 2\sum_{j=0}^{2}\sum_{i=0}^{1} B_i^1(u)B_j^2(v)\,(P_{i+1,j} - P_{i,j})\,,$$

$$X_v = 2\sum_{i=0}^{2}\sum_{j=0}^{1} B_j^1(v)\,B_i^2(u)\,\Delta^{0,1}\,P_{i,j} = 2\sum_{i=0}^{2}\sum_{j=0}^{1} B_j^1(v)\,B_i^2(u)\,(P_{i,j+1} - P_{i,j})\,,$$

$$X_{uu} = 2\sum_{j=0}^{2} B_j^2(v)\,\Delta^{2,0}\,P_{i,j} = 2\sum_{j=0}^{2} B_j^2(v)\,(P_{2,j} - 2P_{1,j} + P_{0,j}); i = 0\,,$$

$$X_{vv} = 2\sum_{i=0}^{2} B_i^2(u)\,\Delta^{0,2}\,P_{i,j} = 2\sum_{i=0}^{2} B_i^2(u)\,(P_{i,2} - 2P_{i,1} + P_{i,0}); j = 0\,,$$

$$X_{uv} = 4\sum_{i=0}^{1}\sum_{j=0}^{1} B_i^1(u)\,B_j^1(v)\,\Delta^{1,1}\,P_{i,j} = 4\sum_{i=0}^{1}\sum_{j=0}^{1} B_i^1(u)\,B_j^1(v)[(P_{i+1,j+1} - P_{i+1,j}) - (P_{i,j+1} - P_{i,j})]\,, \quad (20)$$

and the normal vector is

$$\mathbf{n}(u,v) = 4\sum_{j=0}^{2}\sum_{i=0}^{1}\sum_{k=0}^{2}\sum_{l=0}^{1} B_i^1(u)\,B_k^2(u)\,B_l^1(v)\,B_j^2(v)\;\nabla(P_{i,j} \times P_{k,l}), \quad (21)$$

whereby, we abbreviated $\Delta^{1,0}P_{i,j} \times \Delta^{0,1}\,P_{i,j}$ by $\nabla(P_{i,j} \times P_{k,l})$.

Now, we are able to calculate the first and the second fundamental forms and then, by applying Eq. (18) we get the lines of curvature of this patch.

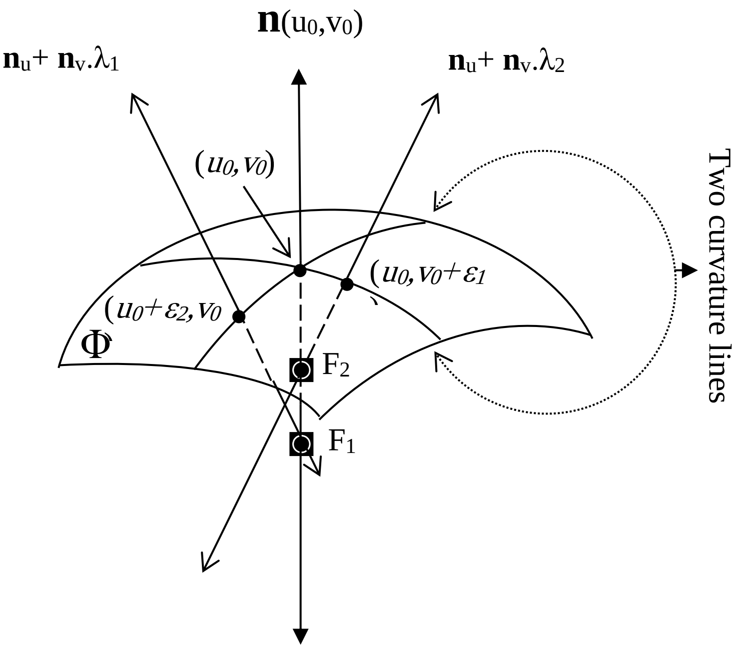

Figure 4. Curvature lines of the patch and two focal points, F_1 and F_2.

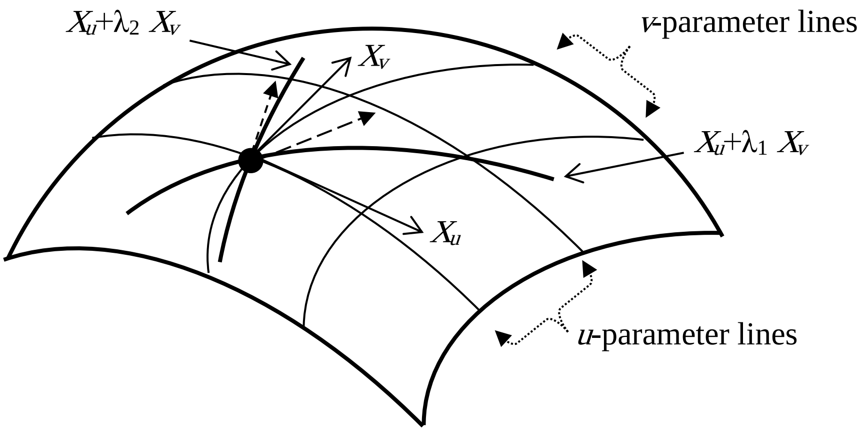

Figure 5. Parameter lines on the patch.

Note that, along the principal curvature lines, the normals fulfill developable surfaces, therefore, $\boldsymbol{n}$ intersects $\boldsymbol{n}_1$ (in focal point F_1) and $\boldsymbol{n}_2$ (in focal point F_2), whereby

$$\mathbf{n}_1 := \mathbf{n}_u + \lambda_1 \mathbf{n}_v \ , \ \mathbf{n}_2 := \mathbf{n}_u + \lambda_2 \mathbf{n}_v \ , \tag{22}$$

and

$$\{\mathbf{n}(u,v), \mathbf{n}_u(u,v), \mathbf{n}_v(u,v)\}, \tag{23}$$

span a plane which is tangent to Klein model M_4^2 and the intersection between M_4^2 and focal plane leads to join of lines, and we have

$$\left(\mathbf{n}(u,v)+\eta\,\mathbf{n}_u(u,v)+\beta\,\mathbf{n}_v(u,v)\right)\in M_4^2,$$

so, we get

$$\left\langle \mathbf{n}+\eta\,\mathbf{n}_u+\beta\,\mathbf{n}_v\ ,\mathbf{n}+\eta\,\mathbf{n}_u+\beta\,\mathbf{n}_v \right\rangle=0$$

$$\Rightarrow\langle\mathbf{n},\mathbf{n}\rangle+2\eta\langle\mathbf{n},\mathbf{n}_u\rangle+2\beta\langle\mathbf{n},\mathbf{n}_v\rangle+2\eta\beta\langle\mathbf{n}_u,\mathbf{n}_v\rangle+\eta^2\langle\mathbf{n}_u,\mathbf{n}_u\rangle+\beta^2\langle\mathbf{n}_v,\mathbf{n}_v\rangle=0, \qquad (24)$$

because $\langle\mathbf{n},\mathbf{n}\rangle=0\ \ for\ all\ u,v$ we get

$$2\eta\beta\langle\mathbf{n}_u,\mathbf{n}_v\rangle+\eta^2\langle\mathbf{n}_u,\mathbf{n}_u\rangle+\beta^2\langle\mathbf{n}_v,\mathbf{n}_v\rangle=0$$

$$\Rightarrow\frac{\beta^2}{\eta^2}\langle\mathbf{n}_v,\mathbf{n}_v\rangle+2\frac{\beta}{\eta}\langle\mathbf{n}_u,\mathbf{n}_v\rangle+\langle\mathbf{n}_u,\mathbf{n}_u\rangle=0\,. \qquad (25)$$

Setting $\frac{\beta}{\eta}=\frac{dv}{du}=\lambda$, we can get the two direction coefficients (λ_1,λ_2) which are known as torsal directions.

Using Eqs. (4), we get *E*, *F* and *G*, that follows:

$$\begin{aligned}E&=\left(2\sum_{j=0}^{2}\sum_{i=0}^{1}B_i^1(u)B_j^2(v)\,(P_{i+1,j}-P_{i,j})\right)^2\\&=4\big((1-u)(1-v)^2(P_{10}-P_{00})+2v(1-u)(1-v)(P_{11}-P_{01})\\&\qquad+v^2(1-u)(P_{12}-P_{02})+u(1-v)^2(P_{20}-P_{10})\\&\qquad\qquad+2uv(1-v)(P_{21}-P_{11})+uv^2(P_{22}-P_{12})\big)^2\ ,\end{aligned}$$

$$\begin{aligned}F&=4\left(\sum_{j=0}^{2}\sum_{i=0}^{1}B_i^1(u)B_j^2(v)\,(P_{i+1,j}-P_{i,j})\right)\left(\sum_{i=0}^{2}\sum_{j=0}^{1}B_j^1(v)B_i^2(u)\,(P_{i,j+1}-P_{i,j})\right)\\&=4\big((1-u)(1-v)^2(P_{10}-P_{00})+2v(1-u)(1-v)(P_{11}-P_{01})\\&\qquad+v^2(1-u)(P_{12}-P_{02})+u(1-v)^2(P_{20}-P_{10})\\&\qquad\qquad+2uv(1-v)(P_{21}-P_{11})+uv^2(P_{22}-P_{12})\big)\\&\quad.\big((1-v)(1-u)^2(P_{01}-P_{00})+2u(1-u)(1-v)(P_{11}-P_{10})\\&\qquad+u^2(1-v)(P_{21}-P_{20})+v(1-u)^2(P_{02}-P_{01})\\&\qquad\qquad+2uv(1-u)(P_{12}-P_{11})+u^2v\ \ (P_{22}-P_{21})\big),\end{aligned}$$

and

$$G = \left(2\sum_{i=0}^{2}\sum_{j=0}^{1} B_j^1(v)B_i^2(u)(P_{i,j+1} - P_{i,j}) \right)^2$$

$$= 4\big((1-v)(1-u)^2(P_{01} - P_{00}) + 2u(1-u)(1-v)(P_{11} - P_{10})$$

$$+ u^2(1-v)(P_{21} - P_{20}) + v(1-u)^2(P_{02} - P_{01})$$

$$+ 2uv(1-u)(P_{12} - P_{11}) + u^2v\ (P_{22} - P_{21})\big)^2 \ .$$

Also, from Eqs. (10), we get *L, M* and *N* that follows:

$$L = \boldsymbol{n}\left(2\sum_{j=0}^{2} B_j^2(v)(P_{2,j} - 2P_{1,j} + P_{0,j} \right)$$

$$= 2\boldsymbol{n}\big((1-v)^2(P_{00} - 2P_{10} + P_{20}) + 2v(1-v)(P_{01} - 2P_{11} + P_{21}) + v^2(P_{02} - 2P_{12} + P_{22})\big),$$

$$M = \boldsymbol{n}\left(4\sum_{i=0}^{1}\sum_{j=0}^{1} B_i^1(u)B_j^1(v)(P_{i+1,j+1} - P_{i+1,j} - P_{i,j+1} + P_{i,j} \right)$$

$$= 4n\big((1-u)(1-v)(P_{00} - P_{01} - P_{10} - P_{11}) + v(1-u)(P_{01} - P_{02} - P_{11} + P_{12})$$

$$+ u(1-v)(P_{10} - P_{11} - P_{20} + P_{21}) + uv(P_{11} - P_{12} - P_{21} + P_{22})\big),$$

and

$$N = \boldsymbol{n}\left(2\sum_{i=0}^{2} B_i^2(u)(P_{i,2} - 2P_{i,1} + P_{i,0} \right)$$

$$= 2\boldsymbol{n}\big((1-u)^2(P_{00} - 2P_{01} + P_{02}) + 2u(1-u)(P_{10} - 2P_{11} + P_{12}) + u^2(P_{20} - 2P_{21} + P_{22})\big).$$

After using Eq. (18) we can calculate the two direction coefficients λ_1, λ_2.

Now, we calculate the two direction coefficients λ_1, λ_2 at the four corner points of the patch, $P_{00}, P_{02}, P_{20}\ and\ P_{22}$, where at these points the parameters u and v are $(0,0), (0,1), (1,0)\ \ and\ \ (1,1)$, respectively.

At the first, let us calculate *E, F, G, L, M* and *N* at these points as follows:

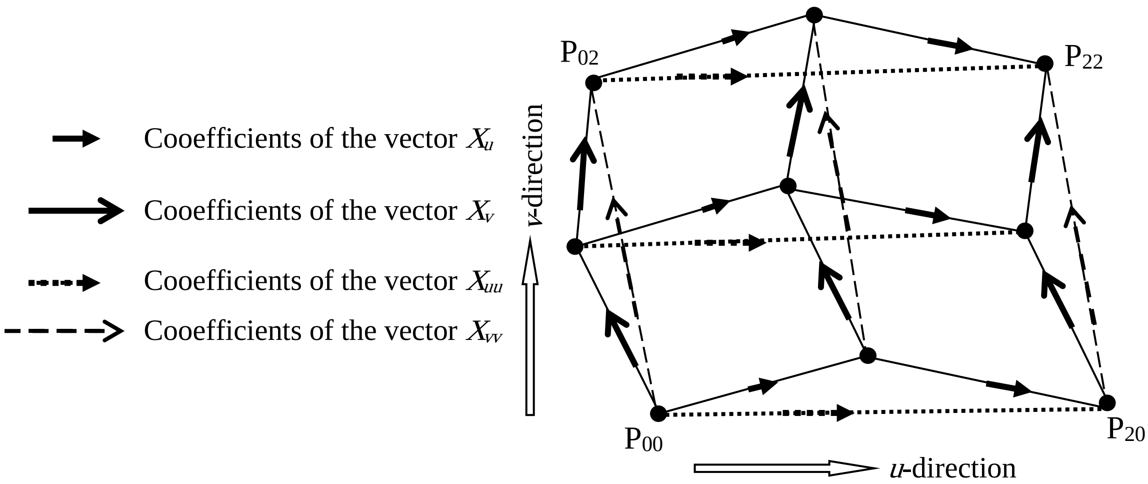

Figure 6. The coefficients of the partial derivatives.

Setting, $\Delta P_{i\,j\,k\,l} = (P_{i\,j} - P_{k\,l})$, to simplify our calculations.

- At the point P_{00} where $(u,v) = (0,0)$:

$$E = 4(\Delta P_{1000})^2,\ F = 4(\Delta P_{0100})(\Delta P_{1000})\ and\ G = 4(\Delta P_{0100})^2,$$
$$\boldsymbol{n} = 4(\Delta P_{1000} \times \Delta P_{0100}),$$
$$L = 8\det(\Delta P_{1000}, \Delta P_{0100}, \Delta P_{2010}),$$
$$M = 16\det(\Delta P_{1000}, \Delta P_{0100}, \Delta P_{1101}) \text{ or } M = 16\det(\Delta P_{1000}, \Delta P_{0100}, \Delta P_{1110}),$$
$$N = 8\det(\Delta P_{1000}, \Delta P_{0100}, \Delta P_{0201}).$$

- At the point P_{02} where $(u,v) = (0,1)$:

$$E = 4(\Delta P_{1202})^2,\ F = 4(\Delta P_{0201})(\Delta P_{1202})\ and\ G = 4(\Delta P_{0201})^2.$$
$$\boldsymbol{n} = 4(\Delta P_{1202} \times \Delta P_{0201}),$$
$$L = 8\det(\Delta P_{1202}, \Delta P_{0201}, \Delta P_{2212}),$$
$$M = 16\det(\Delta P_{1202}, \Delta P_{0201}, \Delta P_{0111}) \text{ or } M = 16\det(\Delta P_{1202}, \Delta P_{0201}, \Delta P_{1211}),$$
$$N = 8\det(\Delta P_{1202}, \Delta P_{0201}, \Delta P_{0001}).$$

- At the point P_{20} where $(u,v) = (1,0)$:

$$E = 4(\Delta P_{2010})^2,\ F = 4(\Delta P_{2010})(\Delta P_{2120})\ and\ G = 4(\Delta P_{2120})^2.$$
$$\boldsymbol{n} = 4(\Delta P_{2010} \times \Delta P_{2120}),$$
$$L = 8\det(\Delta P_{2010}, \Delta P_{2120}, \Delta P_{0010}),$$
$$M = 16\det(\Delta P_{2010}, \Delta P_{2120}, \Delta P_{1011}) \text{ or } M = 16\det(\Delta P_{2010}, \Delta P_{2120}, \Delta P_{2111}),$$
$$N = 8\det(\Delta P_{2010}, \Delta P_{2120}, \Delta P_{2221}).$$

- At the point P_{22} where $(u,v)=(1,1)$:

$$E = 4(\Delta P_{2212})^2\ ,\ F = 4(\Delta P_{2212})(\Delta P_{2221})\ and\ \ G = 4(\Delta P_{2221})^2\ .$$
$$\boldsymbol{n} = 4(\Delta P_{2212}\times \Delta P_{2221}),$$
$$L = 8\det(\Delta P_{2212},\Delta P_{2221},\Delta P_{0212}),$$
$$M = 16\det(\Delta P_{2212},\Delta P_{2221},\Delta P_{1121})\ \text{or}\ M = 16\det(\Delta P_{2212},\Delta P_{2221},\Delta P_{1112}),$$
$$N = 8\det(\Delta P_{2212},\Delta P_{2221},\Delta P_{2021}).$$

If we use the following symbol, we can get simple equations:

$$^{e\,f\,g\,h}\sum\nolimits_{m\,n\,o\,p}^{i\,j\,k\,l}\Delta P = \det(\Delta P_{e\,f\,g\,h},\Delta P_{i\,j\,k\,l},\Delta P_{m\,n\,o\,p}).$$

Now, we are in a position to calculate the quantities λ_1 and λ_2 which are define directions in u,v-plane at each control corner point as follows:

<u>At the point</u> P_{00}:

$$\lambda_1,\lambda_2 =$$
$$\Big\{32\Big[(\Delta P_{0100})^2\ {}^{1000}\sum\nolimits_{2010}^{0100}\Delta P - (\Delta P_{1000})^2\ {}^{1000}\sum\nolimits_{0201}^{0100}\Delta P\Big] \pm \Big[\Big(32((\Delta P_{1000})^2\ {}^{1000}\sum\nolimits_{0201}^{0100}\Delta P$$
$$-(\Delta P_{0100})^2\ {}^{1000}\sum\nolimits_{2010}^{0100}\Delta P)\Big)^2 - 4\Big((32)^2\Big((\Delta P_{0100})(\Delta P_{1000})\ {}^{1000}\sum\nolimits_{0201}^{0100}\Delta P - 2(\Delta P_{0100})^2\ {}^{1000}\sum\nolimits_{1101}^{0100}\Delta P\Big)$$
$$\Big(2(\Delta P_{1000})^2\ {}^{1000}\sum\nolimits_{1101}^{0100}\Delta P - (\Delta P_{1000})(\Delta P_{0100})\ {}^{1000}\sum\nolimits_{2010}^{0100}\Delta P\Big)\Big)\Big]^{1/2}\Big\}$$
$$\Big/ 64\Big[(\Delta P_{0100})(\Delta P_{1000})\ {}^{1000}\sum\nolimits_{0201}^{0100}\Delta P - 2(\Delta P_{0100})^2\ {}^{1000}\sum\nolimits_{1101}^{0100}\Delta P\Big].$$

<u>At the point</u> P_{02}:

$$\lambda_1,\lambda_2 =$$
$$\Big\{32\Big[(\Delta P_{0201})^2\ {}^{1202}\sum\nolimits_{2212}^{0201}\Delta P - (\Delta P_{1202})^2\ {}^{1202}\sum\nolimits_{0001}^{0201}\Delta P\Big] \pm \Big[\Big(32((\Delta P_{1202})^2\ {}^{1202}\sum\nolimits_{0001}^{0201}\Delta P$$
$$-(\Delta P_{0201})^2\ {}^{1202}\sum\nolimits_{2212}^{0201}\Delta P)\Big)^2 - 4\Big((32)^2\Big((\Delta P_{0201})(\Delta P_{1202})\ {}^{1202}\sum\nolimits_{0001}^{0201}\Delta P - 2(\Delta P_{0201})^2\ {}^{1202}\sum\nolimits_{0111}^{0201}\Delta P\Big)$$
$$\Big(2(\Delta P_{1202})^2\ {}^{1202}\sum\nolimits_{0111}^{0201}\Delta P - (\Delta P_{0201})(\Delta P_{1202})\ {}^{1202}\sum\nolimits_{2212}^{0201}\Delta P\Big)\Big)\Big]^{1/2}\Big\}$$
$$\Big/ 64\Big[(\Delta P_{0201})(\Delta P_{1202})\ {}^{1202}\sum\nolimits_{0001}^{0201}\Delta P - 2(\Delta P_{0201})^2\ {}^{1202}\sum\nolimits_{0111}^{0201}\Delta P\Big].$$

<u>At the point</u> P_{20}:

$$\lambda_1, \lambda_2 = \Big\{ 32\Big[(\Delta P_{2120})^2 \ {}^{2010}\!\sum_{0010}^{2120} \Delta P - (\Delta P_{2010})^2 \ {}^{2010}\!\sum_{1011}^{2120} \Delta P\Big] \pm \Big[\Big(32((\Delta P_{2010})^2 \ {}^{2010}\!\sum_{1011}^{2120} \Delta P - (\Delta P_{2120})^2 \ {}^{2010}\!\sum_{0010}^{2120} \Delta P)\Big)^2 - 4\Big((32)^2\Big((\Delta P_{2010})(\Delta P_{2120}) \ {}^{2010}\!\sum_{2221}^{2120} \Delta P - 2(\Delta P_{2120})^2 \ {}^{2010}\!\sum_{1011}^{2120} \Delta P\Big) \Big(2(\Delta P_{2010})^2 \ {}^{2010}\!\sum_{1011}^{2120} \Delta P - (\Delta P_{2010})(\Delta P_{2120}) \ {}^{2010}\!\sum_{0010}^{2120} \Delta P\Big)\Big)\Big]^{1/2} \Big\} \Big/ 64\Big[(\Delta P_{2010})(\Delta P_{2120}) \ {}^{2010}\!\sum_{2221}^{2120} \Delta P - 2(\Delta P_{2120})^2 \ {}^{2010}\!\sum_{1011}^{2120} \Delta P\Big].$$

<u>At the point</u> P_{22}:

$$\lambda_1, \lambda_2 = \Big\{ 32\Big[(\Delta P_{2221})^2 \ {}^{2212}\!\sum_{0212}^{2221} \Delta P - (\Delta P_{2212})^2 \ {}^{2212}\!\sum_{1121}^{2221} \Delta P\Big] \pm \Big[\Big(32((\Delta P_{2212})^2 \ {}^{2212}\!\sum_{1121}^{2221} \Delta P - (\Delta P_{2221})^2 \ {}^{2212}\!\sum_{0212}^{2221} \Delta P)\Big)^2 - 4\Big((32)^2\Big((\Delta P_{2212})(\Delta P_{2221}) \ {}^{2212}\!\sum_{2021}^{2221} \Delta P - 2(\Delta P_{2221})^2 \ {}^{2212}\!\sum_{1121}^{2221} \Delta P\Big) \Big(2(\Delta P_{2212})^2 \ {}^{2212}\!\sum_{1121}^{2221} \Delta P - (\Delta P_{2212})(\Delta P_{2221}) \ {}^{2212}\!\sum_{0212}^{2221} \Delta P\Big)\Big)\Big]^{1/2} \Big\} \Big/ 64\Big[(\Delta P_{2212})(\Delta P_{2221}) \ {}^{2212}\!\sum_{2021}^{2221} \Delta P - 2(\Delta P_{2221})^2 \ {}^{2212}\!\sum_{1121}^{2221} \Delta P\Big].$$

Also, from Eq. (16) we can calculate the two principal curvatures at each corner control point and then, we get two radii of curvature at these points.

The following figures illustrate the coefficient vectors of two direction formulas at the four corner control points of the bi-quadratic TP Bézier patch.

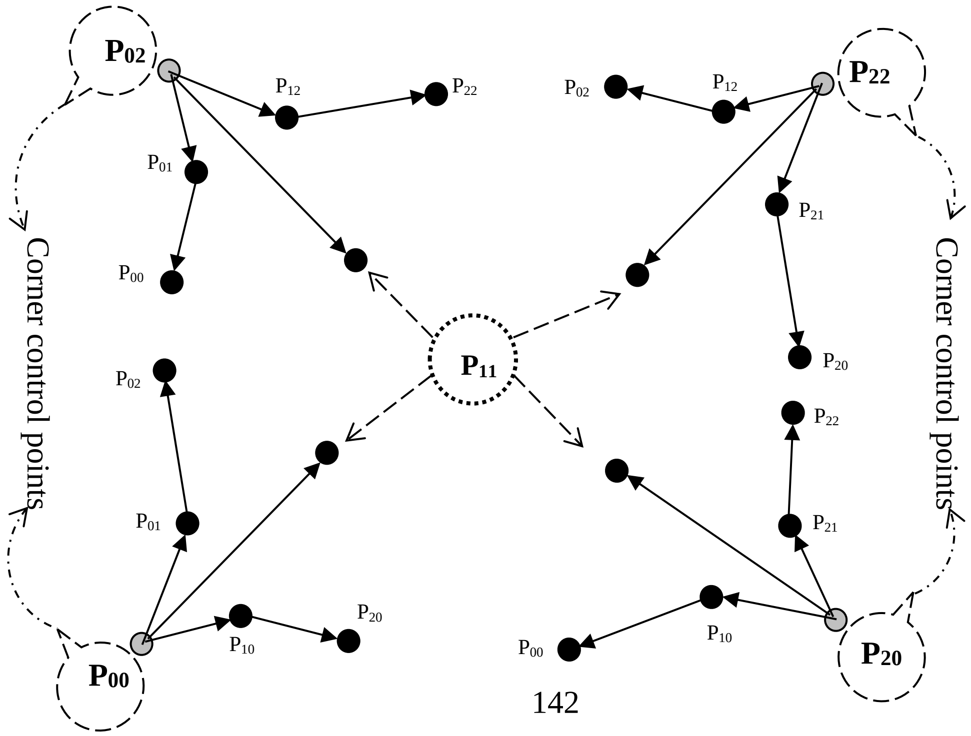

Figure 7. Coefficient vectors of the first and second fundamental quantities.

The normal vector to the TP Bèzier Patch is denoted by

$$N = mn\sum_{j=0}^{n}\sum_{i=0}^{m-1}\sum_{k=0}^{m}\sum_{l=0}^{n-1} B_i^{m-1}(u)\, B_k^{m}(u)\, B_j^{n-1}(v) B_l^{n}(v)\ \ (\Delta^{1,0}P_{i,j} \times \Delta^{0,1}P_{k,l}).$$

For the biquadratic Patch; *m=n=2*, the normal vector becomes

$$N = 4\sum_{j=0}^{2}\sum_{i=0}^{1}\sum_{k=0}^{2}\sum_{l=0}^{1} B_i^{1}(u)\, B_k^{2}(u)\, B_j^{1}(v) B_l^{2}(v)\ \ (\Delta^{1,0}P_{i,j} \times \Delta^{0,1}P_{k,l}).$$

Taking derivatives, we get

$$N_u = \frac{\partial N}{\partial u} = 4\sum_{j=0}^{2}\sum_{l=0}^{1}\left[\frac{\partial}{\partial u}\sum_{i=0}^{1}\sum_{k=0}^{2} B_i^{1}(u) B_k^{2}(u)\ (\Delta^{1,0}P_{i,j} \times \Delta^{0,1}P_{k,l})\right] B_j^{2}(v) B_l^{1}(v)\ .$$

The bracketed term depends only on u, and we can apply the formula for the derivative of a Bèzier curve:

$$N_u = 8\sum_{j=0}^{2}\sum_{l=0}^{1}\sum_{k=0}^{1} B_k^{1}(u) B_l^{1}(v) B_j^{2}(v)\Big[\big(\Delta^{2,0}P_{0,j} \times \Delta^{0,1}P_{k,l}\big) + \big(\Delta^{1,0}P_{0,j} \times \Delta^{1,1}P_{k,l}\big)\Big],$$

by the same way, we have

$$N_v = 8\sum_{i=0}^{1}\sum_{k=0}^{2}\sum_{j=0}^{1} B_k^{2}(u) B_i^{1}(u) B_j^{1}(v)\Big[\big(\Delta^{1,1}P_{i,j} \times \Delta^{0,1}P_{k,0}\big) + \big(\Delta^{1,0}P_{i,j} \times \Delta^{0,2}P_{k,0}\big)\Big],$$

where

$$\Delta^{1,0}\,P_{i,j} = P_{i+1,j} - P_{i,j}\quad,\quad \Delta^{0,1}\,P_{i,j} = P_{i,j+1} - P_{i,j}\ \ ,$$
$$\Delta^{0,2}\,P_{i,j} = \Delta^{0,1}\,P_{i,j+1} - \Delta^{0,1}P_{i,j}\ ,\ \Delta^{2,0}\,P_{i,j} = \Delta^{1,0}\,P_{i+1,j} - \Delta^{1,0}P_{i,j}\ \ ,$$
$$\Delta^{1,1}\,P_{i,j} = (P_{i+1,j+1} - P_{i+1,j}) - (P_{i,j+1} - P_{i,j}).$$

Let us calculate N_u and N_v at the four corner points, we get

<u>At the point</u> *P_{00}* where $(u,v) = (0,0)$:

$$N_u = 8\{[(P_{10} - P_{00})\times(P_{11} - P_{10})] + [(P_{20} - P_{10})\times(P_{01} - P_{00})] - 2[(P_{10} - P_{00})\times(P_{01} - P_{00})]\},$$

$$N_v = 8\{[(P_{11}-P_{10})\times(P_{01}-P_{00})]-[(P_{10}-P_{00})\times(P_{01}-P_{00})]-[(P_{02}-P_{01})\times(P_{10}-P_{00})]\}.$$

<u>At the point</u> *P02* where $(u,v)=(0,1)$:

$$N_u = 8\{[(P_{22}-P_{12})\times(P_{02}-P_{01})]+[(P_{12}-P_{02})\times(P_{12}-P_{11})]-2[(P_{12}-P_{02})\times(P_{02}-P_{01})]\},$$

$$N_v = 8\{[(P_{12}-P_{11})\times(P_{01}-P_{00})]+[(P_{11}-P_{01})\times(P_{02}-P_{01})]-[(P_{02}-P_{01})\times(P_{01}-P_{00})] -[(P_{11}-P_{01})\times(P_{01}-P_{00})]\}.$$

<u>At the point</u> *P20* where $(u,v)=(1,0)$:

$$N_u = 8\{[(P_{20}-P_{10})\times(P_{11}-P_{10})]+[(P_{10}-P_{00})\times(P_{21}-P_{20})]-2[(P_{10}-P_{00})\times(P_{11}-P_{10})]\},$$

$$N_v = 8\{[(P_{20}-P_{10})\times(P_{22}-P_{21})]-[(P_{11}-P_{10})\times(P_{21}-P_{20})]-[(P_{20}-P_{10})\times(P_{21}-P_{20})]\}.$$

<u>At the point</u> *P22* where $(u,v)=(1,1)$:

$$N_u = 8\{[(P_{22}-P_{12})\times(P_{12}-P_{11})]+[(P_{12}-P_{02})\times(P_{22}-P_{21})]-2[(P_{12}-P_{02})\times(P_{12}-P_{11})]\},$$

$$N_v = 8\{[(P_{22}-P_{21})\times(P_{21}-P_{20})]+[(P_{21}-P_{11})\times(P_{22}-P_{21})]-[(P_{12}-P_{11})\times(P_{21}-P_{20})] -[(P_{21}-P_{11})\times(P_{21}-P_{20})]\}.$$

In the above equations, using $\Delta^{1,0} P_{i,j} = P_{i+1,j} - P_{i,j}$, $\Delta^{0,1} P_{i,j} = P_{i,j+1} - P_{i,j}$.

We have the normal vectors N_u and N_v in simple forms as follows:

<u>At the point</u> *P00* where $(u,v)=(0,0)$:

$$N_u = 8\{(\Delta^{1,0} P_{00}\times\Delta^{0,1} P_{10})+(\Delta^{1,0} P_{10}\times\Delta^{0,1} P_{00})-2(\Delta^{1,0} P_{00}\times\Delta^{0,1} P_{00})\},$$

$$N_v = 8\{(\Delta^{0,1} P_{10}\times\Delta^{0,1} P_{00})-(\Delta^{1,0} P_{00}\times\Delta^{0,1} P_{00})-(\Delta^{0,1} P_{01}\times\Delta^{1,0} P_{00})\}.$$

<u>At the point</u> *P02* where $(u,v)=(0,1)$:

$$N_u = 8\{(\Delta^{1,0} P_{12}\times\Delta^{0,1} P_{01})+(\Delta^{1,0} P_{02}\times\Delta^{0,1} P_{11})-2(\Delta^{1,0} P_{02}\times\Delta^{0,1} P_{01})\},$$

$$N_v = 8\{(\Delta^{0,1} P_{11}\times\Delta^{0,1} P_{00})+(\Delta^{1,0} P_{01}\times\Delta^{0,1} P_{01})-(\Delta^{0,1} P_{01}\times\Delta^{0,1} P_{00})-(\Delta^{1,0} P_{01}\times\Delta^{0,1} P_{00})\}.$$

<u>At the point</u> *P20* where $(u,v)=(1,0)$:

$$N_u = 8\left\{\left(\Delta^{1,0} P_{10} \times \Delta^{0,1} P_{10}\right) + \left(\Delta^{1,0} P_{00} \times \Delta^{0,1} P_{20}\right) - 2\left(\Delta^{1,0} P_{00} \times \Delta^{0,1} P_{10}\right)\right\},$$

$$N_v = 8\left\{\left(\Delta^{1,0} P_{10} \times \Delta^{0,1} P_{21}\right) - \left(\Delta^{0,1} P_{10} \times \Delta^{0,1} P_{20}\right) - \left(\Delta^{1,0} P_{10} \times \Delta^{0,1} P_{20}\right)\right\}.$$

<u>At the point</u> P_{22} where $(u,v) = (1,1)$:

$$N_u = 8\left\{\left(\Delta^{1,0} P_{12} \times \Delta^{0,1} P_{11}\right) + \left(\Delta^{1,0} P_{02} \times \Delta^{0,1} P_{21}\right) - 2\left(\Delta^{1,0} P_{02} \times \Delta^{0,1} P_{11}\right)\right\},$$

$$N_v = 8\left\{\left(\Delta^{0,1} P_{21} \times \Delta^{0,1} P_{20}\right) + \left(\Delta^{1,0} P_{11} \times \Delta^{0,1} P_{21}\right) - \left(\Delta^{0,1} P_{11} \times \Delta^{0,1} P_{20}\right) - \left(\Delta^{1,0} P_{11} \times \Delta^{0,1} P_{20}\right)\right\}.$$

Now, at each corner point on the patch, we can calculate the two focal points;

$$F_1 = N_u + \lambda_1 N_v \qquad \text{and} \qquad F_2 = N_u + \lambda_2 N_v .$$

Note that, λ_1 and λ_2 depend on the choice corner control point, so we have

<u>At the point</u> P_{00} where $(u,v) = (0,0)$:

$$F_1 = 8\left\{\left[\left(\Delta^{1,0} P_{00} \times \Delta^{0,1} P_{10}\right) + \left(\Delta^{1,0} P_{10} \times \Delta^{0,1} P_{00}\right) - 2\left(\Delta^{1,0} P_{00} \times \Delta^{0,1} P_{00}\right)\right] + \lambda_1 \left[\left(\Delta^{0,1} P_{10} \times \Delta^{0,1} P_{00}\right) - \left(\Delta^{1,0} P_{00} \times \Delta^{0,1} P_{00}\right) - \left(\Delta^{0,1} P_{01} \times \Delta^{1,0} P_{00}\right)\right]\right\},$$

$$F_2 = 8\left\{\left[\left(\Delta^{1,0} P_{00} \times \Delta^{0,1} P_{10}\right) + \left(\Delta^{1,0} P_{10} \times \Delta^{0,1} P_{00}\right) - 2\left(\Delta^{1,0} P_{00} \times \Delta^{0,1} P_{00}\right)\right] + \lambda_2 \left[\left(\Delta^{0,1} P_{10} \times \Delta^{0,1} P_{00}\right) - \left(\Delta^{1,0} P_{00} \times \Delta^{0,1} P_{00}\right) - \left(\Delta^{0,1} P_{01} \times \Delta^{1,0} P_{00}\right)\right]\right\}.$$

<u>At the point</u> P_{02} where $(u,v) = (0,1)$:

$$F_1 = 8\left\{\left[\left(\Delta^{1,0} P_{12} \times \Delta^{0,1} P_{01}\right) + \left(\Delta^{1,0} P_{02} \times \Delta^{0,1} P_{11}\right) - 2\left(\Delta^{1,0} P_{02} \times \Delta^{0,1} P_{01}\right)\right] + \lambda_1 \left[\left(\Delta^{0,1} P_{11} \times \Delta^{0,1} P_{00}\right) + \left(\Delta^{1,0} P_{01} \times \Delta^{0,1} P_{01}\right) - \left(\Delta^{0,1} P_{01} \times \Delta^{0,1} P_{00}\right) - \left(\Delta^{1,0} P_{01} \times \Delta^{0,1} P_{00}\right)\right]\right\},$$

$$F_2 = 8\left\{\left[\left(\Delta^{1,0} P_{12} \times \Delta^{0,1} P_{01}\right) + \left(\Delta^{1,0} P_{02} \times \Delta^{0,1} P_{11}\right) - 2\left(\Delta^{1,0} P_{02} \times \Delta^{0,1} P_{01}\right)\right] + \lambda_2 \left[\left(\Delta^{0,1} P_{11} \times \Delta^{0,1} P_{00}\right) + \left(\Delta^{1,0} P_{01} \times \Delta^{0,1} P_{01}\right) - \left(\Delta^{0,1} P_{01} \times \Delta^{0,1} P_{00}\right) - \left(\Delta^{1,0} P_{01} \times \Delta^{0,1} P_{00}\right)\right]\right\}.$$

<u>At the point</u> P_{20} where $(u,v) = (1,0)$:

$$F_1 = 8\left\{\left[\left(\Delta^{1,0} P_{10} \times \Delta^{0,1} P_{10}\right) + \left(\Delta^{1,0} P_{00} \times \Delta^{0,1} P_{20}\right) - 2\left(\Delta^{1,0} P_{00} \times \Delta^{0,1} P_{10}\right)\right] + \lambda_1 \left[\left(\Delta^{1,0} P_{10} \times \Delta^{0,1} P_{21}\right) - \left(\Delta^{0,1} P_{10} \times \Delta^{0,1} P_{20}\right) - \left(\Delta^{1,0} P_{10} \times \Delta^{0,1} P_{20}\right)\right]\right\},$$

$$F_2 = 8\left\{\left[\left(\Delta^{1,0} P_{10} \times \Delta^{0,1} P_{10}\right) + \left(\Delta^{1,0} P_{00} \times \Delta^{0,1} P_{20}\right) - 2\left(\Delta^{1,0} P_{00} \times \Delta^{0,1} P_{10}\right)\right] + \lambda_2 \left[\left(\Delta^{1,0} P_{10} \times \Delta^{0,1} P_{21}\right) - \left(\Delta^{0,1} P_{10} \times \Delta^{0,1} P_{20}\right) - \left(\Delta^{1,0} P_{10} \times \Delta^{0,1} P_{20}\right)\right]\right\}.$$

<u>At the point</u> P_{22} where $(u,v) = (1,1)$:

$$F_1 = 8\{[(\Delta^{1,0}P_{12} \times \Delta^{0,1}P_{11}) + (\Delta^{1,0}P_{02} \times \Delta^{0,1}P_{21}) - 2(\Delta^{1,0}P_{02} \times \Delta^{0,1}P_{11})] + \lambda_1 \\ [(\Delta^{0,1}P_{21} \times \Delta^{0,1}P_{20}) + (\Delta^{1,0}P_{11} \times \Delta^{0,1}P_{21}) - (\Delta^{0,1}P_{11} \times \Delta^{0,1}P_{20}) - (\Delta^{1,0}P_{11} \times \Delta^{0,1}P_{20})]\},$$

$$F_2 = 8\{[(\Delta^{1,0}P_{12} \times \Delta^{0,1}P_{11}) + (\Delta^{1,0}P_{02} \times \Delta^{0,1}P_{21}) - 2(\Delta^{1,0}P_{02} \times \Delta^{0,1}P_{11})] + \lambda_2 \\ [(\Delta^{0,1}P_{21} \times \Delta^{0,1}P_{20}) + (\Delta^{1,0}P_{11} \times \Delta^{0,1}P_{21}) - (\Delta^{0,1}P_{11} \times \Delta^{0,1}P_{20}) - (\Delta^{1,0}P_{11} \times \Delta^{0,1}P_{20})]\}.$$

4. Computational example

We have the control points for the biquadratic TP Bèzier patch, are denoted by the following matrix:

$$P_{i,j} = \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} (1,-1,0) & (2,-1,1) & (1,3,2) \\ (-3,2,1) & (0,2,1) & (1,0,3) \\ (-1,3,1) & (-2,2,1) & (3,-1,2) \end{bmatrix}; \; i,j = 0,1,2.$$

The first and second fundamental quantities, two principal directions, two principal curvatures and two radii of curvature at the four corner points are:

<u>At the point</u> *P_{00}* :

$$E = 104, F = -12, G = 8, L = 88, M = 144 \; and \; N = 112.$$

So, we get

$$\lambda_1 = 5.541, \; \lambda_2 = -1.158, \; \kappa(\lambda_1) = 23.646, \; \kappa(\lambda_2) = -0.669, \rho_1 = 0.042, \rho_2 = -1.495.$$

<u>At the point</u> *P_{02}* :

$$E = 40, F = -44, G = 72, L = -80, M = -176 \; and \; N = 80.$$

So, we get

$$\lambda_1 = 0.691, \; \lambda_2 = -1.669, \; \kappa(\lambda_1) = -21.004, \; \kappa(\lambda_2) = 1.885, \rho_1 = -0.0428, \rho_2 = 0.530.$$

<u>At the point</u> *P_{20}* :

$$E = 20, F = -12, G = 8, L = 8, M = 0 \; and \; N = -8.$$

So, we get

$$\lambda_1 = 1.761, \; \lambda_2 = 0.565, \; \kappa(\lambda_1) = -6.605, \; \kappa(\lambda_2) = 0.606, \rho_1 = -0.151, \rho_2 = 1.651.$$

<u>At the point</u> *P_{22}* :

$$E = 24, F = 48, G = 140, L = -160, M = -128 \; and \; N = -88.$$

So, we get

$$\lambda_1 = -1.20, \; \lambda_2 = -0.280, \; \kappa(\lambda_1) = 0.186, \; \kappa(\lambda_2) = -11.761, \rho_1 = 5.391, \rho_2 = -0.085.$$

Also, we get the normal vectors in two directions *u* and *v* at the four corner control points as follows:

At the point P_{00} ; $N_u = (-40,-72,-32)$ *and* $N_v = (-32,-40,-80)$.

At the point P_{02} ; $N_u = (104,16,128)$ *and* $N_v = (-48,-8,32)$.

At the point P_{20} ; $N_u = (8,-56,176)$ *and* $N_v = (8,-16,-56)$.

At the point P_{22} ; $N_u = (32,-16,48)$ *and* $N_v = (-8,24,-8)$.

Finally, we are in a position to calculate the two focal points F1 and F2 at each corner control point as follows:

At the point P_{00} : $F_1 = (-217.312, -293.64, -475.28)$, $F_2 = (-2.944, -25.68, 60.64)$.

At the point P_{02} : $F_1 = (70.832, 10.472, 150.112)$, $F_2 = (184.112, 29.352, 74.592)$.

At the point P_{20} :

$$F_1 = (22.08, -84.16, 68.44), \ F_2 = (12.52, -56.04, 135.36).$$

At the point P_{22} :

$$F_1 = (41.6, -44.8, 57.6), \ F_2 = (34.24, -22.72, 50.24).$$

The following figure shows this patch

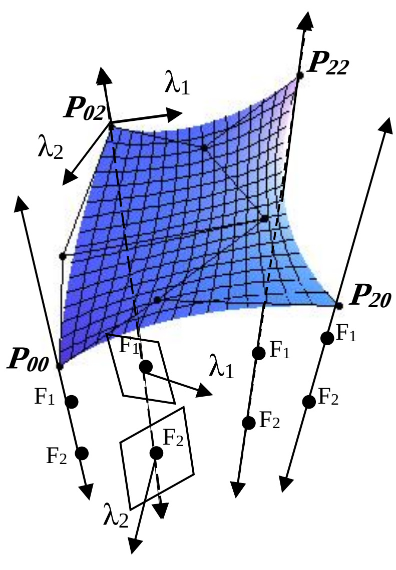

Figure 8. Bi-quadratic TP- *Bèzier* patch with normals and their focal points at the four corner control points.

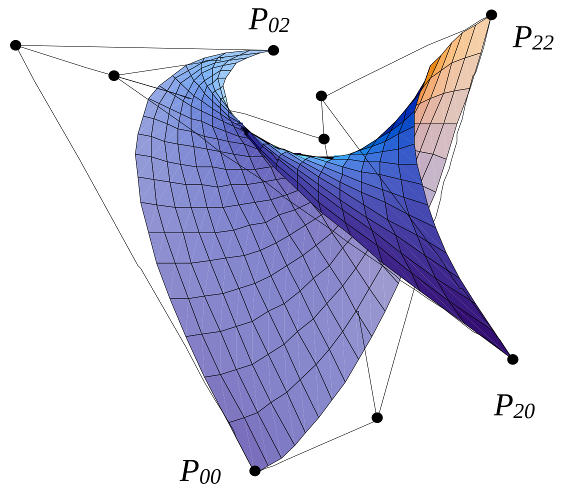

Figure 9. Bi-quadraticTP Bèzier patch with four corner control points.

5. Conclusion

In five-dimensional real projective space (P^5), a new approach for tensor product (TP) Bézier patch representation has been introduced. Derivatives, normal vectors of Bézier patches and some of geometric properties of these patches have been discussed. Finally, a computational example of the two focal points of a normal of this patch is given and plotted.

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