Mixed Vector Equilibrium Problems with Fuzzy Mappings

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Abstract

In this paper, we study fuzzy generalized mixed vector equilibrium problem and fuzzy mixed vector equilibrium problem. We prove existence results for fuzzy generalized mixed vector equilibrium problem and fuzzy mixed vector equilibrium problem by using some basic tools as KKM theory and Maximal element lemma. We provide sufficient conditions that ensure the existence of the solution of these problems. The results presented in this paper generalize, improve and unify the previously known results in this area. An example is given.

keywords

Vector equilibrium problem, KKM theory, Maximal element lemma, Fuzzy setting, Existence, AMS Subject Classification: 47J22; 47J25; 47S40

اتزان المتجه الغامض المعمم المدمج ومشكلة حالة اتزانه

الملخص: في هذا البحث, نحن ندرس مسألة حالة اتزان المتجه الغامض المعمم المدمج ومشكلة حالة اتزان المتجه الغامض المدمج, نحن نثبت وجود نتائج لمسألة حالة اتزان المتجه الغامض المعمم المدمج ومسألة حالة اتزان المتجه الغامض المدمج باستخدام بعض الطرق الأساسية مثل نظرية(ك م م) ونظرية العنصر الأقصى.

كما نوفر شروط كافية لضمان وجود حلول لهذه المسألة. النتائج التي تقدم في هذا البحث معممة وتحسن وتوحد النتائج المعروفة السابقة في هذا المجال, وأيضًا أُعطيت أمثلة.

1 Introduction

Equilibrium problems have been the subject of considerable research and have profound contributions in large variety of problems of practical interest arising in nonlinear analysis, optimization, economics, finance and game theory. It includes many mathematical problems as particular cases such as mathematical programming problems, complementarity problems, variational inequality problems, fixed point problems, minimax inequality problems and Nash equilibrium problems in noncooperative games, see for example, [8, 14, 18, 30]. The foundation of (scalar) equilibrium theory has been laid down by Ky Fan [17], his minimax inequality still being considered one of the most notable results in this field. The classical scalar equilibrium problem in [17], described by a bifunction $f : K \times K \to R$ consists in finding $x \in K$ such that

$$f(x,y) \geq 0, \ \ \forall\, y \in K.$$

It was Blum and Oettli [8], who used the term equilibrium problem for the first time. Starting with the pioneering work of Giannessi [18], several extensions of the scalar equilibrium problem to the vector case have been considered. These vector equilibrium problems, much like their scalar counterpart, offer a unified framework for treating vector optimization, vector variational inequalities or cone saddle point problems, see for examples [2, 3, 4, 6, 19, 21]. Let X and Z be locally convex Hausdorff topological vector spaces, $K \subseteq X$ be a nonempty set and let $C \subseteq Z$ be a convex and pointed cone. Assume that the interior of the cone C, denoted by $intC$, is nonempty and consider the mapping $f : K \times K \to Z$. The vector equilibrium problem, consists in finding $x \in K$, such that

$$f(x,y) \notin -intC, \ \forall\, y \in K.$$

Because of its applications and interests, vector equilibrium problems have been studied by many authors in different directions using variant techniques, see, for example, [5, 20, 29]. The concept of fuzzy set theory was introduced by Zadeh [36] and penetrates almost all branches of mathematics. Fuzzy set theory have been applied to many fields including information science, artificial intelligence, computer science, management science and control engineering, etc., see [32, 37].

In 1981, Heilpern [22] proved a fixed point theorem for fuzzy contraction mapping which is fuzzy analogue of Nadlar's fixed point theorem for set-valued mapping. In 1989, Chang and Zhu [9] introduced the concept of variational inequalities for fuzzy mappings in abstract spaces and investigated the existence problem for solutions of some class of variational inequalities for fuzzy mappings. Since then several classes of variational inequalities (inclusions) and vector variational inequalities with fuzzy mappings were considered by Chang and Huang [10], Ding and Park [15], Chang and Salahuddin [12], Anastassio and Salahuddin [1], Lan and Verma [26] and Lee et al. [28]. Recently, Huang and Lan [23] considered non linear equations with fuzzy mappings in fuzzy normed spaces. Very recently, Rahaman and Ahmad [31] studied fuzzy vector equilibrium problem. They proved some existence results by using particular forms of results of Kim and Lee [25] and Tarafdar [34]. Motivated by the research mentioned above and ongoing research in this direction, in this paper, we study some mixed vector equilibrium problems in fuzzy setting.

2 Preliminaries

Now, we mention some definitions, notations and conclusions which are needed in the sequel.
A mapping F from E into the collection $\mathscr{F}(E)$ of all fuzzy sets of E is called fuzzy mapping. If $F: E \to \mathscr{F}(E)$ is a fuzzy mapping, then $F(x), x \in E$ (denoted by F_x, in the sequel) is a fuzzy set in $\mathscr{F}(E)$, which is a function from E to $[0,1]$. For each $y \in E, F_x(y)$ is the degree of membership of y in F_x. Let $A \in \mathscr{F}(E)$ and $\alpha \in [0,1]$, then the set
$A_\alpha = \{x \in E : A(x) \geq \alpha\}$ is called an α-cut set of A.
Definition 2.1 A mapping $f: K \to Z$ is said to be convex, if for any $x_1, x_2 \in K$ and $t \in [0,1]$,

$$f(tx_1 + (1-t)x_2) \leq_{C(x)} tf(x_1) + (1-t)f(x_2),$$

which is equivalent to

$$tf(x_1) + (1-t)f(x_2) - f(tx_1 + (1-t)x_2) \in C(x).$$

Definition 2.2 (see, [35]) Let X, Y be two topological vector spaces, $T: X \to 2^Y$ be a set-valued mapping and $T^{-1}(y) = \{x \in X : y \in T(x)\}$. Then

(i) T is said to be upper semi-continuous, if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, then there exists an open neighbourhood U of x in X such that $T(u) \subset V$, for each $u \in U$.

(ii) T is said to be closed, if for any net $\{x_\alpha\}$ in X such that $x_\alpha \to x$ and any net $\{y_\alpha\}$ in Y such that $y_\alpha \to y$ and $y_\alpha \in T(x_\alpha)$ for any α, we have $y \in T(x)$.

(iii) T is said to have a closed graph, if the graph of $T, Gr(T) = \{(x,y) \in X \times Y : y \in T(x)\}$ is closed in $X \times Y$.

Definition 2.3 (see, [27]) Let X, Y be topological spaces and $T: X \to \mathscr{F}(Y)$ be a fuzzy mapping. T is said to have fuzzy set-valued, if $T_x(y)$ is upper semi-continuous on $X \times Y$ as a real ordinary function.
Lemma 2.1 If A is a closed subset of a topological space X, then the characteristic function $\mathscr{X}_A$ of A is an upper semi-continuous real-valued function.

Lemma 2.2 (see, [24]) Let K be a non-empty closed convex subset of a real Hausdorff topological vector space X, E be a non-empty closed convex subset of a real Hausdorff topological vector space Y and $a : X \to [0,1]$ be a lower semi-continuous function. Let $T : K \to \mathscr{F}(E)$ be a fuzzy mapping with $(Tx)_{a(x)} \neq \emptyset, \forall\, x \in X$ and $\bar{T} : K \to 2^E$ be a multi-function defined by $\bar{T}(x) = (Tx)_{a(x)}$. If T is a closed set-valued mapping, then $\bar{T}$ is a closed multi-function.

Definition 2.4 (see, [13]) Let K be a convex subset of a topological vector space E and Z be a topological vector space. Let $C : K \to 2^Z$ be a set-valued mapping. For any given finite subset $\{x_1, x_2, \cdots, x_n\}$ of K and any $x = \sum_{i=1}^n t_i x_i$ with $t_i \geq 0$ for $i = 1, 2, \cdots, n$ and $\sum_{i=1}^n t_i = 1$,

(i) a single valued mapping $M : K \times K \to Z$ is said to be vector 0-diagonally convex in the second variable, if

$$\sum_{i=1}^{n} t_i M(x, x_i) \notin -intC(x);$$

(ii) a set-valued mapping $M : K \times K \to 2^Z$ is said to be generalized vector 0-diagonally convex in the second variable, if

$$\sum_{i=1}^{n} t_i u_i \notin -intC(x),\ \forall\, u_i \in M(x, x_i).$$

Lemma 2.3 (see, [33]) Let X, Y be two topological spaces and $T : X \to 2^Y$ be an upper semi-continuous set-valued mapping with compact values. Suppose $\{x_\alpha\}$ is a net in X such that $x_\alpha \to x_0$. If $y_\alpha \in T(x_\alpha)$ for each α, then there exists $y_0 \in T(x_0)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y_0$.

Lemma 2.4 (see, [7]) Let X and Y be two topological spaces. If $T : X \to 2^Y$ is an upper semi-continuous set-valued mapping with closed values, then T is closed.

Lemma 2.5 (see, [11], Maximal Element Lemma) Let X be a non-empty convex subset of a topological vector space E and $T : X \to 2^X$ be a set-valued mapping satisfying the following conditions:

(i) for each $x \in X, x \notin CoT(x)$ and for each $y \in X, T^{-1}(y)$ is open-valued in X;

(ii) there exist a non-empty compact subset A of X and a non-empty convex subset B of X such that

$$Co(T(x)) \cap B \neq \emptyset,\ \forall\, x \in X \setminus A.$$

Then there exists $x_0 \in X$ such that $T(x_0) = \emptyset$.

Definition 2.5 (KKM Mapping) Let K be a subset of a topological vector space X. A set-valued mapping $A : K \to 2^X$ is said to be KKM-mapping, if for each finite subset $\{x_1, x_2, ..., x_n\}$ of K, $Co\{x_1, x_2, \cdots, x_n\} \subseteq \bigcup_{i=1}^{n} A(x_i)$, where $Co\{x_1, x_2, \cdots, x_n\}$ denotes the convex hull of $\{x_1, x_2, \cdots, x_n\}$.

Lemma 2.6 (see, [16], KKM-Fan Theorem) Let K be a subset of a Hausdorff topological vector space X and let $A : K \to 2^X$ be a KKM mapping. If for each $x \in K, A(x)$ is closed and if for at least one point $x \in K, A(x)$ is compact, then $\bigcap_{x \in K} A(x) \neq \emptyset$.

Let Y be a locally convex Hausdorff topological vector space and X be a Hausdorff topological vector space, $L(X,Y)$, the space of all continuous linear operators from X into Y be a locally convex space being equipped with σ-topology. Let K be a non-empty convex subset of a Hausdorff topological vector space $X, C : K \to 2^Y$ be a set-valued mapping such that $intC(x) \neq \emptyset$, $\forall\ x \in K$. Let $T : L(X,Y) \times L(X,Y) \times L(X,Y) \to 2^{L(X,Y)}, f : K \times K \to 2^Y$, $g : K \to K$ and $\phi : X \times X \to 2^Y$ be the vector-valued mappings. Let $P,Q,R : K \to \mathscr{F}(L(X,Y))$ be the fuzzy mappings and $a,b,c : K \to [0,1]$ are the functions. The partial order relation $\leq_{C(x)}$ in Y with the convex cone $C(x)$ is defined as

$$y_1 \leq_{C(x)} y_2 \Leftrightarrow y_2 - y_1 \in C(x),\ \forall y_1, y_2 \in Y.$$

In this paper, we consider the following fuzzy generalized mixed vector equilibrium problem: Find $x \in K, u \in (Px)_{a(x)}, v \in (Qx)_{b(x)}$ and $w \in (Rx)_{c(x)}$ such that

$$f(x,y) + \langle T(u,v,w), y - g(x)\rangle + \phi(x,y) - \phi(x,x) \notin -intC(x),\ \forall\ y \in K \tag{1}$$

and the fuzzy mixed vector equilibrium problem: Find $x \in K, u \in (Px)_{a(x)}, v \in (Qx)_{b(x)}$ and $w \in (Rx)_{c(x)}$ such that

$$f(x,y) + \langle T(u,v,w), y - x\rangle + \phi(x,y) - \phi(x,x) \notin -intC(x),\ \forall\ y \in K. \tag{2}$$

In support of problem (1), we construct the following example.

Example 2.1 Let $X = Y = K = C = [0,1]$, we define the fuzzy mappings $P,Q,R : K \to \mathscr{F}(L(X,Y))$, for all $u,v,w \in [0,1]$ as follows:
$P_x(u) = \begin{cases} 0, & \text{if } x \in [0,\frac{1}{2}); \\ \frac{x+u}{2}, & \text{if } x \in [\frac{1}{2},1], \end{cases}$ $Q_x(v) = \begin{cases} 0, & \text{if } x \in [0,\frac{1}{3}); \\ \frac{x+v}{3}, & \text{if } x \in [\frac{1}{3},1], \end{cases}$ $R_x(w) = \begin{cases} 0, & \text{if } x \in [0,\frac{1}{4}); \\ \frac{x+w}{4}, & \text{if } x \in [\frac{1}{4},1], \end{cases}$ and the mapping $a,b,c : K \to [0,1]$ as follows:
$a(x) = \begin{cases} 0, & \text{if } x \in [0,\frac{1}{2}); \\ \frac{x}{2}, & \text{if } x \in [\frac{1}{2},1], \end{cases}$ $b(x) = \begin{cases} 0, & \text{if } x \in [0,\frac{1}{3}); \\ \frac{x}{3}, & \text{if } x \in [\frac{1}{3},1], \end{cases}$ $c(x) = \begin{cases} 0, & \text{if } x \in [0,\frac{1}{4}); \\ \frac{x}{4}, & \text{if } x \in [\frac{1}{4},1]. \end{cases}$
Clearly, $P_x(u) \geq a(x), Q_x(v) \geq b(x)$ and $R_x(w) \geq c(x), \forall x \in K$, i.e., $u \in (P_x)_{a(x)}, v \in (Q_x)_{b(x)}$ and $w \in (R_x)_{c(x)}$.

Now, we define the mappings $f : K \times K \to Y, \phi : X \times X \to Y, g : K \to K$ and $T : L(X,Y) \times L(X,Y) \times L(X,Y) \to Y$ as follows:
$f(x,y) = \frac{x+y}{2}, \phi(x,y) = \frac{x-y}{2}, g(x) = x^2$ and $T(u,v,w) = \begin{cases} 0, & \text{if } x \in [0,\frac{1}{2}); \\ \frac{1}{2}\sin^2(uvw), & \text{if } x \in [\frac{1}{2},1]. \end{cases}$ Then, one can verify that
$f(x,y) + \langle T(u,v,w), y - g(x)\rangle + \phi(x,y) - \phi(x,x) \notin -intC(x), \forall y \in K$. Thus, fuzzy generalized mixed vector equilibrium problem (1) is satisfied.

3 Existence Results

In the following theorem, we prove the existence of solution for fuzzy generalized mixed vector equilibrium problem (1).

Theorem 3.1 Let Y be a locally convex Hausdorff topological vector space, K be a non-empty convex subset of a Hausdorff topological vector space X. Let $P,Q,R : K \to \mathscr{F}(L(X,Y))$ be the fuzzy mappings and $\mathscr{P},\mathscr{Q},\mathscr{R} : K \to 2^{L(X,Y)}$ be upper semi-continuous set-valued mappings with non-empty compact values induced by fuzzy mappings P,Q and R, respectively, i.e.,

$$\mathscr{P}(x) = (Px)_{a(x)}, \mathscr{Q}(x) = (Qx)_{b(x)}, \mathscr{R}(x) = (Rx)_{c(x)}, \forall x \in K.$$

Let $T : L(X,Y) \times L(X,Y) \times L(X,Y) \to 2^{L(X,Y)}, f : K \times K \to 2^Y, \phi : X \times X \to 2^Y$ be the set-valued mappings and $g : K \to K$ be the single-valued mapping such that the following conditions hold:

(i) f is generalized vector 0-diagonally convex and affine in the second argument;

(ii) g is continuous mapping with $x - g(x) = 0,\ \forall\, x \in K$;

(iii) ϕ is affine in the second argument;

(iv) $C : K \to 2^Y$ be a convex set-valued mapping and $Y \setminus \{-intC(x)\}$ is upper semi-continuous ;

(v) for all $y \in K,\ f(\cdot,y) + \langle T(\cdot,\cdot,\cdot), y - g(\cdot)\rangle + \phi(\cdot,y) + \phi(\cdot,\cdot)$ is upper semi-continuous with compact values;

(vi) there exists a non-empty compact subset A of K and a non-empty compact convex subset D of K such that for all $x \in K \setminus A$, there exists $\bar{y} \in D$ such that

$$f(x,\bar{y}) + \langle T(u,v,w), \bar{y} - g(x)\rangle + \phi(x,\bar{y}) - \phi(x,x) \in -intC(x),$$

$$\forall\, u \in \mathscr{P}(x) = (Px)_{a(x)}, v \in \mathscr{Q}(x) = (Qx)_{B(x)}, w \in \mathscr{R}(x) = (Rx)_{c(x)},\ \forall\, x \in K.$$

Then the fuzzy generalized mixed vector equilibrium problem (1) has a solution.

Proof. We define a set-valued mapping $M : K \to 2^K$ by

$$M(x) = \{y \in K : f(x,y) + \langle T(u,v,w), y - g(x)\rangle + \phi(x,y) - \phi(x,x) \in -intC(x),$$

$$\forall\, u \in \mathscr{P}(x) = (Px)_{a(x)}, v \in \mathscr{Q}(x) = (Qx)_{b(x)}, w \in \mathscr{R}(x) = (Rx)_{c(x)}\},\ \forall\, x \in K.$$

Now, we have to show that $x \notin coM(x), \forall\, x \in K$. Suppose to the contrary that there exists $\bar{x} \in K$ such that $\bar{x} \in coM(\bar{x})$, then there exists a finite set $\{y_1, y_2, \cdots, y_n\}$ such that $\bar{x} \in co\{y_1, y_2, \cdots, y_n\}$, we have

$$f(\bar{x},y_i)+\langle T(u,v,w),y_i-g(\bar{x})\rangle+\phi(\bar{x},y_i)-\phi(\bar{x},\bar{x})\in -intC(\bar{x}),\ i=1,2,\cdots,n,$$

$$\forall\ u\in\mathscr{P}(\bar{x})=(P\bar{x})_{a(\bar{x})},v\in\mathscr{Q}(\bar{x})=(Q\bar{x})_{b(\bar{x})},w\in\mathscr{R}(\bar{x})=(R\bar{x})_{c(\bar{x})}.$$

Since $intC(\bar{x})$ is a convex set, for each $i=1,2,\cdots,n$ with $\sum_{i=1}^{n}t_i=1,t_i\in[0,1],\bar{x}=\sum_{i=1}^{n}t_iy_i$, then it follows from affinity of f,ϕ in the second argument and by condition (ii) that

$$\begin{aligned}
f(\bar{x},\textstyle\sum_{i=1}^{n}t_iy_i) \quad + &\quad \langle T(u,v,w),\textstyle\sum_{i=1}^{n}t_iy_i-g(\bar{x})\rangle+\phi(\bar{x},\textstyle\sum_{i=1}^{n}t_iy_i)-\phi(\bar{x},\bar{x})\\
= &\quad \textstyle\sum_{i=1}^{n}t_if(\bar{x},y_i)+\langle T(u,v,w),\bar{x}-g(\bar{x})\rangle+\phi(\bar{x},\bar{x})-\phi(\bar{x},\bar{x})\\
= &\quad \textstyle\sum_{i=1}^{n}t_if(\bar{x},y_i)\in -intC(\bar{x}),\\
= &\quad \textstyle\sum_{i=1}^{n}t_iu_i\in -intC(\bar{x}),\forall u_i\in f(\bar{x},y_i),i=1,2,\cdots,n,
\end{aligned}$$

which is a contradiction to the generalized vector 0-diagonal convexity of f in the second argument and hence $x\notin coM(x),\ \forall\ x\in K$.

Next, we prove that for each $y\in K,M^{-1}(y)$ is an open set. To prove this, we need to show that the complement $[M^{-1}(y)]^c$ of $M^{-1}(y)$ is closed, i.e.,

$$[M^{-1}(y)]^c=\{x\in K:\{f(x,y)+\langle T(u,v,w),y-g(x)\rangle+\phi(x,y)-\phi(x,x)\}\cap Y\setminus\{-intC(x)\}\neq\emptyset,$$

$$\forall\ u\in\mathscr{P}(x)=(Px)_{a(x)},v\in\mathscr{Q}(x)=(Qx)_{b(x)},w\in\mathscr{R}(x)=(Rx)_{c(x)}\}$$

is a closed set in K. Let $\{x_\alpha\}$ be a sequence in $[M^{-1}(y)]^c$ such that $x_\alpha\to x_0$, then there exist $u_\alpha\in\mathscr{P}(x_\alpha)=(Px_\alpha)_{a(x_\alpha)},v_\alpha\in\mathscr{Q}(x_\alpha)=(Qx_\alpha)_{b(x_\alpha)},w_\alpha\in\mathscr{R}(x_\alpha)=(Rx_\alpha)_{c(x_\alpha)}$ such that

$$\{f(x_\alpha,y)+\langle T(u_\alpha,v_\alpha,w_\alpha),y-g(x_\alpha)\rangle+\phi(x_\alpha,y)-\phi(x_\alpha,x_\alpha)\}\cap Y\setminus\{-intC(x_\alpha)\}\neq\emptyset.$$

Since $\mathscr{P},\mathscr{Q},\mathscr{R}:K\to 2^{L(X,Y)}$ are compact valued upper semi-continuous mappings induced by the fuzzy mappings $P,Q,R:K\to\mathscr{F}(L(X,Y))$, respectively, then by Lemma 2.3, $\{u_\alpha\},\{v_\alpha\}$ and $\{w_\alpha\}$ have the subsequences $\{u_{\alpha_n}\},\{v_{\alpha_n}\}$ and $\{w_{\alpha_n}\}$, respectively, such that $u_{\alpha_n}\to u_0,v_{\alpha_n}\to v_0$ and $w_{\alpha_n}\to w_0$ and $u_0\in\mathscr{P}(x_0)=(Px_0)_{a(x_0)},v_0\in\mathscr{Q}(x_0)=(Qx_0)_{b(x_0)},w_0\in\mathscr{R}(x_0)=(Rx_0)_{c(x_0)}$. Suppose that

$$\nu_\alpha\in\{f(x_\alpha,y)+\langle T(u_\alpha,v_\alpha,w_\alpha),y-g(x_\alpha)\rangle+\phi(x_\alpha,y)-\phi(x_\alpha,x_\alpha)\}\cap Y\setminus\{-intC(x_\alpha)\}.$$

Since $f(\cdot,y)+\langle N(\cdot,\cdot,\cdot),y-g(\cdot)\rangle+\phi(\cdot,y)-\phi(\cdot,\cdot)\cap Y\setminus\{-intC(x_\alpha)\}$ is upper semi-continuous and compact values, it follows from Lemma 2.3 that there exists $\nu_0\in f(x_0,y)+\langle T(u_0,v_0,w_0),y-g(x_0)\rangle+\phi(x_0,y)-f(x_0,x_0)$ and a subsequence $\{\nu_{\alpha_n}\}$ of $\{\nu_\alpha\}$ such that $\nu_{\alpha_n}\to\nu_0$. Since $Y\setminus\{-intC(x)\}$ is an upper semi-continuous with closed values, hence by Lemma 2.1, $\nu_0\in Y\setminus$

$\{-intC(x_0)\}$, thus we have

$$\{f(x_0,y)+\langle T(u_0,v_0,w_0),y-g(x_0)\rangle+\phi(x_0,y)-f(x_0,x_0)\}\cap Y\setminus\{-intC(x_0)\}\neq\emptyset.$$

Hence $[M^{-1}(y)]^c$ is a closed subset in K and therefore $M^{-1}(y)$ is an open set for all $y\in K$. It follows from the condition (vi) of the theorem that for each $x\in K\setminus A$, there exists $\bar{y}\in D$ such that $x\in intM^{-1}(\bar{y})$. Thus, all the conditions of the Lemma 2.5 are satisfied, hence there exists $x\in K$ such that $M(x)=\emptyset$, i.e., there exists $x\in K$, $u\in\mathscr{P}(x)=(Px)_{a(x)}, v\in\mathscr{Q}(x)=(Qx)_{b(x)}, w\in\mathscr{R}(x)=(Rx)_{c(x)}$ such that

$$f(x,y)+\langle T(u,v,w),y-g(x)\rangle+\phi(x,y)-\phi(x,x)\notin -intC(x),\ \forall\, y\in K.$$

This completes the proof.

In the following theorem, we prove the existence of solution for the fuzzy mixed vector equilibrium problem (2), which is obtained by taking $g\equiv I$, the identity mapping in Theorem 3.1.

Theorem 3.2 Let Y be a locally convex Hausdorff topological vector space, K be a non-empty convex subset of a Hausdorff topological vector space X. Let $P,Q,R:K\to\mathscr{F}(L(X,Y))$ be the fuzzy mappings and $\mathscr{P},\mathscr{Q},\mathscr{R}:K\to 2^{L(X,Y)}$ be upper semi-continuous set-valued mappings with non-empty compact values induced by fuzzy mappings P,Q and R, respectively, i.e.,

$$\mathscr{P}(x)=(Px)_{a(x)},\mathscr{Q}(x)=(Qx)_{b(x)},\mathscr{R}(x)=(Rx)_{c(x)}.$$

Let $T:L(X,Y)\times L(X,Y)\times L(X,Y)\to 2^{L(X,Y)}, f:K\times K\to 2^Y$ and $\phi:X\times X\to 2^Y$ be the set-valued mappings such that the following conditions hold:

(i) f is generalized vector 0-diagonally convex and affine in the second argument;

(iii) ϕ is affine in the second argument;

(iii) $C:K\to 2^Y$ be a convex set-valued mapping and $Y\setminus\{-intC(x)\}$ is upper semi-continuous;

(iv) for all $y\in K$, $f(\cdot,y)+\langle T(\cdot,\cdot,\cdot),y-(\cdot)\rangle+\phi(\cdot,y)-\phi(x,x)$ is upper semi-continuous with compact values;

(v) there exists a non-empty compact subset A of K and a non-empty compact convex subset D of K such that for all $x\in K\setminus A$, there exists $\bar{y}\in D$ such that

$$f(x,\bar{y})+\langle T(u,v,w),\bar{y}-x\rangle+\phi(x,\bar{y})-\phi(x,x)\in -intC(x),$$

$$\forall\, u\in\mathscr{P}(x)=(Px)_{a(x)},v\in\mathscr{Q}(x)=(Qx)_{B(x)},w\in\mathscr{R}(x)=(Rx)_{c(x)},\ \forall\, x\in K.$$

Then the fuzzy mixed vector equilibrium problem (2) has a solution.

Proof. We define a set-valued mapping $M : K \to 2^K$ by

$$M(x) = \{y \in K : f(x,y) + \langle T(u,v,w), y-x\rangle + \phi(x,y) - \phi(x,x) \in -intC(x),$$

$$\forall\, u \in \mathscr{P}(x) = (Px)_{a(x)}, v \in \mathscr{Q}(x) = (Qx)_{B(x)}, w \in \mathscr{R}(x) = (Rx)_{c(x)}\},\ \forall\, x \in K.$$

Now, we have to show that $x \notin coM(x), \forall\, x \in K$. Suppose to the contrary that there exists $\bar{x} \in K$ such that $\bar{x} \in coM(\bar{x})$. Then there exists a finite set $\{y_1, y_2, \cdots, y_n\}$ such that $\bar{x} \in co\{y_1, y_2, \cdots, y_n\}$, then we have

$$f(\bar{x}, y_i) + \langle T(u,v,w), y_i - \bar{x}\rangle + \phi(\bar{x}, y_i) - \phi(\bar{x}, \bar{x}) \in -intC(\bar{x}),\ i = 1, 2, \cdots, n,$$

$$\forall\, u \in \mathscr{P}(x) = (Px)_{a(x)}, v \in \mathscr{Q}(x) = (Qx)_{b(x)}, w \in \mathscr{R}(x) = (Rx)_{c(x)}.$$

Since $intC(\bar{x})$ is a convex set, for each $i = 1, 2, \cdots, n$ with $\sum_{i=1}^n t_i = 1, t_i \in [0,1], \bar{x} = \sum_{i=1}^n t_i y_i$, then it follows from the affinity of f and ϕ in the second argument that

$$\begin{aligned}
f(\bar{x}, \textstyle\sum_{i=1}^n t_i y_i) \quad &+ \quad \langle T(u,v,w), \textstyle\sum_{i=1}^n t_i y_i - \bar{x}\rangle + \phi(\bar{x}, \textstyle\sum_{i=1}^n t_i y_i) - \phi(\bar{x}, \bar{x}) \\
&= \quad \textstyle\sum_{i=1}^n t_i f(\bar{x}, y_i) + \langle T(u,v,w), \bar{x} - \bar{x}\rangle + \phi(\bar{x}, \bar{x}) - \phi(\bar{x}, \bar{x}) \\
&= \quad \textstyle\sum_{i=1}^n t_i f(\bar{x}, y_i) \in -intC(\bar{x}), \\
&= \quad \textstyle\sum_{i=1}^n t_i u_i \in -intC(\bar{x}), \forall u_i \in f(\bar{x}, y_i), i = 1, 2, \cdots, n,
\end{aligned}$$

which is a contradiction to generalized vector 0-diagonally convexity assumptions of f and hence $x \notin coM(x),\ \forall\, x \in K$.

Next, we prove that for each $y \in K, M^{-1}(y)$ is an open set. To prove this, we need to show that the complement $[M^{-1}(y)]^c$ of $M^{-1}(y)$ is closed, i.e.,

$$[M^{-1}(y)]^c = \{x \in K : \{f(x,y) + \langle T(u,v,w), y-x\rangle + \phi(x,y) - \phi(x,x)\} \cap Y \setminus \{-intC(x)\} \neq \emptyset,$$

$$\forall\, u \in \mathscr{P}(x) = (Px)_{a(x)}, v \in \mathscr{Q}(x) = (Qx)_{b(x)}, w \in \mathscr{R}(x) = (Rx)_{c(x)}\}$$

is a closed set in K. Let $\{x_\alpha\}$ be a sequence in $[M^{-1}(y)]^c$ such that $x_\alpha \to x_0$. Then there exist $u_\alpha \in \mathscr{P}(x_\alpha) = (Px_\alpha)_{a(x_\alpha)}, v_\alpha \in \mathscr{Q}(x_\alpha) = (Qx_\alpha)_{b(x_\alpha)}, w_\alpha \in \mathscr{R}(x_\alpha) = (Rx_\alpha)_{c(x_\alpha)}$ such that

$$\{f(x_\alpha, y) + \langle T(u_\alpha, v_\alpha, w_\alpha), y - x_\alpha\rangle + \phi(x_\alpha, y) - \phi(x_\alpha, x_\alpha)\} \cap Y \setminus \{-intC(x_\alpha)\} \neq \emptyset.$$

Since $\mathscr{P}, \mathscr{Q}, \mathscr{R} : K \to 2^{L(X,Y)}$ are compact valued upper semi-continuous mappings induced by the fuzzy mappings $P, Q, R : K \to \mathscr{F}(L(X,Y))$, respectively, then by Lemma 2.3, $\{u_\alpha\}, \{v_\alpha\}$ and $\{w_\alpha\}$ have the subsequences $\{u_{\alpha_n}\}, \{v_{\alpha_n}\}$ and $\{w_{\alpha_n}\}$, respectively, such that $u_{\alpha_n} \to u_0, v_{\alpha_n} \to v_0$

and $w_{\alpha_n} \to w_0$ and $u_0 \in \mathscr{P}(x_0) = (Px_0)_{a(x_0)}, v_0 \in \mathscr{Q}(x_0) = (Qx_0)_{b(x_0)}, w_0 \in \mathscr{R}(x_0) = (Rx_0)_{c(x_0)}$. Suppose that

$$v_\alpha \in f(x_\alpha, y) + \langle T(u_\alpha, v_\alpha, w_\alpha), y - x_\alpha \rangle + \phi(x_\alpha, y) - \phi(x_\alpha, x_\alpha)\} \cap Y \setminus \{-intC(x_\alpha)\}.$$

Since $f(\cdot, y) + \langle T(\cdot,\cdot,\cdot), y - (\cdot)\rangle + \phi(\cdot, y) - \phi(\cdot,\cdot)$ is upper semi-continuous and compact values, it follows from Lemma 2.3 that there exists $v_0 \in f(x_0, y) + \langle T(u_0, v_0, w_0), y - x_0\rangle + \phi(x_0, y) - \phi(x_0, x_0)$ and a subsequence $\{v_{\alpha_n}\}$ of $\{v_\alpha\}$ such that $v_{\alpha_n} \to v_0$. Since $Y \setminus \{-intC(x)\}$ is an upper semi-continuous with closed values, hence by Lemma 2.1, $v_0 \in Y \setminus \{-intC(x_0)\}$, thus we have

$$\{f(x_0, y) + \langle T(u_0, v_0, w_0), y - x_0\rangle + \phi(x_0, y) - \phi(x_0, x_0)\} \cap Y \setminus \{-intC(x_0)\} \neq \emptyset.$$

Hence $[M^{-1}(y)]^c$ is a closed subset in K and therefore $M^{-1}(y)$ is an open set for all $y \in K$. It follows from the condition (v) of the theorem that for each $x \in K \setminus A$, there exists $\bar{y} \in D$ such that $x \in intM^{-1}(\bar{y})$. Thus, all the conditions of the Lemma 2.5 are satisfied, hence there exists $x \in K$ such that $M(x) = \emptyset$, i.e., there exists $x \in K, u \in \mathscr{P}(x) = (Px)_{a(x)}, v \in \mathscr{Q}(x) = (Qx)_{b(x)}, w \in \mathscr{R}(x) = (Rx)_{c(x)}$ such that

$$f(x, y) + \langle T(u, v, w), y - x\rangle + \phi(x, y) - \phi(x, x) \notin -intC(x),\ \forall\, y \in K.$$

This completes the proof.

4 Competing Interests

The authors declare that there is no competing interests regarding the publication of this article.

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