



On eJU-Algebras: An Extension of JU-Algebras

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Abstract

This article introduces a class of algebraic structures, called eJU-algebras, as a natural extension of JU-algebras using structural hypotheses. The extension arises by replacing classical notion of a constant unit element in JU-algebras with a framework based on non-empty subsets, leading to a broader and more flexible algebraic model. We investigate fundamental properties of eJU-algebras and establish their theoretical foundation by deriving them from traditional JU-algebras. This extension not only enhances the structural richness of JU-algebras but also opens new directions for the development of algebraic systems beyond conventional constraints.

Keywords: eJU-algebras; Minimals; eJU-subalgebras; eJU-ideals.

<https://doi.org/10.63070/jesc.2025.023>

Received 18 July 2025; Revised 16 September 2025; Accepted 24 September 2025.

Available online 12 October 2025.

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1. Introduction

Theory of logical algebras were introduced by mathematician George Boole [1] in 1847 as a formal framework for reasoning and computation. The authors described completeness theorem and many more concepts and theorems on logical algebras. Imai and Iseki [2] in 1966 established some classes of logical algebras; one of them was BCK-algebras. After some time Iseki [3] introduced a different category of logical algebra referred to as BCI-algebras. Iseki has also given their properties as well. Hu and Li [4] in 1983, initiate a class of logical algebras which is BCH-algebras. Komori [5] introduced the idea of BCC-algebras, whereas Dudek [6] in 1992 re-defined the concept of the BCC-algebras, by applying dual form to common definition. Jun et al. [7] in 1998 presented the idea of BH-algebras; this is the generalization of BCK/BCH/BCI-algebra. For further overview about logical algebra one can see [8, 9, 10, 11]. Commutative groups of BCI algebras in specific instances such as Boolean algebras, NM-algebras, residuated lattices, BL-algebras, MTL-algebras, MV-algebras, BE-algebras, Hilbert-algebras and weak-R0 algebras are extensively studied. The researchers focused on the formulation of models of algebras for multiple-valued logics that are not commutative, such algebras are called extended algebraic structures. It inspired us to construct the extended new BCK-algebras called sBCK/sBCI/ eBCK/eBCI-algebras that take into consideration. In this particular case, specifically, a new algebraic structure called sBCK/sBCI/ eBCK/eBCI-algebras are obtained from given BCI/BCK-algebras [12].

JU-algebra is the natural generalization of KU-algebra introduced by Prabpayak and Leerawat [13]. Many authors have conducted extensive research on KU-algebras in a variety of applications including neutrosophic, fuzzy and intuitionistic, rough and soft sense, etc. The notion of JU-algebra was introduced by Ansari, et al. [14]. JU-algebra may rather be thought of as a pseudo KU-algebra. They analyzed the related characteristics of ideals of JU-algebras and p-closure of subsets as well as the p-closure of subset J that is not empty within the JU-algebra X. Romano [15] also discovered various new facts on JU-algebras. Additionally some concepts of JU-algebras, JU-subalgebras, and JU-ideals are also defined in [16]. A few recent studies based on JU algebras structures are presented in [17] and [18].

As generalizations of the concepts of KU/UP-algebras, new concepts of eKU/sKU /eUP/sUP-algebras are proposed. These new extensions of algebras are then investigated for several related features, and it is demonstrated that the axioms of these new extensions are independent [19].

In this work, we originate the notion of an eJU-algebra, extending the framework of JU-algebras. Our focus involves the study of a subset $A \neq \varnothing$, within the eJU-algebras X, exploring the associated properties of subsets of eJU-algebras. Furthermore, we seek into the analysis of eJU-subalgebras, eJU-ideals and Minimals.

In this work, we introduce the concept of eJU-algebras as a meaningful extension of the classical JU-algebra framework. The core of our investigation centers on a non-empty subset $A \neq \emptyset$, within an eJU-algebra X through which we explore a range of structural and algebraic properties intrinsic to such subsets. Building on this foundation, we further examine the internal structure of eJU-algebras by analyzing key components, including eJU-subalgebras, eJU-ideals, and minimal elements. This study not only broadens the theoretical scope of JU-algebras but also provides a deeper understanding of their generalized forms and associated algebraic behavior.

2. Preliminaries

This section includes fundamental definitions of our related topics that help us to understand our main results and to prove the relevant results.

Definition 2.1. A JU-algebra is defined as an algebra $(X, \cdot, 1)$ with a binary operation $\cdot$ satisfying the following:

$$(JU_1) (\gamma \cdot z) \cdot [(z \cdot \kappa) \cdot (\gamma \cdot \kappa)] = 1, \quad \forall \kappa, \gamma, z \in X,$$

$$(JU_2) 1 \cdot \kappa = \kappa, \text{ for } \kappa \in X,$$

$$(JU_3) \kappa \cdot \gamma = \gamma \cdot \kappa = 1 \Rightarrow \kappa = \gamma, \quad \forall \kappa, \gamma \in X.$$

Definition 2.2. A KU-algebra $(X, \cdot, 1)$ with $\cdot$ as single binary operation which satisfies the conditions $\forall \kappa, \gamma, z \in X$.

$$(KU1) (\kappa \cdot \gamma) \cdot [(\gamma \cdot z) \cdot (\kappa \cdot z)] = 1,$$

$$(KU2) \kappa \cdot 1 = 1,$$

$$(KU3) 1 \cdot \kappa = \kappa,$$

$$(KU4) \kappa \cdot \gamma = 1 = \gamma \cdot \kappa \text{ implies } \kappa = \gamma.$$

It is evident from the above definitions that JU-algebras are a specialized class of logical algebraic structures that provides a generalization of foundational framework of KU algebras. We will provide an extension of JU algebras. Extension of algebras means to extend the concept of algebras by giving different hypothesis and generalization to algebras by taking non-empty subsets instead of constants 1.

3. Main Results

We will precisely articulate the concept of eJU-algebras illuminating their algebraic properties and providing relevant examples.

Definition 3.1. An algebra $(X, \cdot, A)$ is called an eJU-algebra, here X is non-empty set, $\cdot$ is single binary operation on X , and its non-empty subset satisfying the following properties:

- (eJU₁) $(\gamma.z).[(z.\kappa).(\gamma.\kappa)] \in A, \forall \kappa, \gamma, z \in X$,
- (eJU₂) $A.\kappa = \{\kappa\}, \forall \kappa \in X$, i.e. $a.x = x, \forall a \in A$,
- (eJU₃) $\kappa.\gamma \in A, \gamma.\kappa \in A$ implies $\kappa = \gamma, \forall \kappa, \gamma \in X \setminus A$.

Definition 3.2. Consider an eJU-algebra $(X, \cdot, A)$ given in above definition. We define a relation " $\leq$ " in X given as $\gamma \leq \kappa$ iff $\kappa.\gamma \in A$, for $\kappa \in X \setminus A$.

Remark 3.1. Note that, whenever $\kappa \leq a$ for any $a \in A$, it implies $a.\kappa = \kappa \in A$ (by Definition 3.2 and eJU₂).

Throughout this article X will represent as eJU-algebra and A will represent as the subset of X with the binary operation $\cdot$ and special element 1 until specified.

Remark 3.2. An eJU-algebra X and its non-empty subset A implies:

Let $\kappa \in A$. Implies $A.\kappa = \{\kappa\}, \forall \kappa \in A$ and $A.\kappa \subseteq A$.

Also, $\bigcup_{\kappa \in A} A.\kappa = A$ and hence $A.A = A$.

Now we will introduce some new lemmas related to our definitions and provide corresponding proofs.

Lemma 3.1. A set $(X, \leq)$ is a POSET if X is an eJU-algebra i.e,

- (eJ₄) $\kappa \leq \kappa$,
- (eJ₅) $\kappa \leq \gamma, \gamma \leq \kappa$ implies $\kappa = \gamma$,
- (eJ₆) $\kappa \leq z, z \leq \gamma$ implies $\kappa \leq \gamma$.

Proof. Taking $\gamma \in A$ and $z \in A$ in eJU₁, we can see that,

$$(A.A).[(A.\kappa).(A.\kappa)] \subseteq A \Rightarrow A.(\kappa.\kappa) \subseteq A \Rightarrow \kappa.\kappa \in A, \text{ by using eJU}_2.$$

It gets, $\kappa \leq \kappa$ and the condition eJ₄ is proved.

As its given: $\kappa \leq \gamma$ and $\gamma \leq \kappa$ which implies $\gamma.\kappa \in A$ and $\kappa.\gamma \in A$ respectively.

Now using eJU₃: $\kappa \leq \gamma$ and $\gamma \leq \kappa \Rightarrow \kappa = \gamma$ and hence eJ₅ is proved.

As we know: $\kappa \leq z$ implies $z.\kappa \in A$ and $z \leq \gamma$ implies $\gamma.z \in A$, showing that $\gamma.\kappa \in A$.

From (eJU_1) ; $(z.\kappa).(\gamma.\kappa) \leq \gamma.z$ because $z.\kappa \in A$, using eJU_2 , implies $(\gamma.z).(\gamma.\kappa) \in A$.

Similarly, $\gamma.\kappa \in A$ because $\gamma.z \in A$.

This implies $\kappa \leq \gamma$, yields eJ_6 is proved. □

Lemma 3.2. An eJU -algebra X satisfies following inequalities $\forall \kappa, \gamma, z \in X$:

$(eJ_7) \kappa \leq \gamma \Rightarrow \gamma.z \leq \kappa.z$,

$(eJ_8) \kappa \leq \gamma \Rightarrow z.\kappa \leq z.\gamma$,

$(eJ_9) (z.\kappa).(\gamma.\kappa) \leq \gamma.z$,

$(eJ_{10}) (\gamma.\kappa).\kappa \leq \gamma$.

Proof. Putting $\kappa = z$ and $z = \kappa$ in eJU_1 $(\gamma.\kappa).[(\kappa.z).(\gamma.z)] \in A$.

Since $\kappa \leq \gamma$ implies $\gamma.\kappa \in A$.

By using eJU_2 : $(\gamma.\kappa).[(\kappa.z).(\gamma.z)] = (\kappa.z).(\gamma.z) \in A \Rightarrow \gamma.z \leq \kappa.z$ means eJ_7 is proved.

As we know that; $\kappa \leq \gamma \Rightarrow \gamma.\kappa \in A$.

Putting $\gamma = z$ and $z = \gamma$ in eJU_1 . $(z.\gamma).[(\gamma.\kappa).(z.\kappa)] \in A$. Since $\gamma.\kappa \in A$, So using eJU_2 :

$(\gamma.\kappa).(z.\kappa) = (z.\kappa) \Rightarrow (z.\gamma).(z.\kappa) \in A \Rightarrow z.\kappa \leq z.\gamma$, eJ_8 is proved.

From eJU_1 , it implies that $[(z.\kappa).(\gamma.\kappa)] \leq (\gamma.z)$, eJ_9 is proved.

Putting $z = \gamma$ and $\gamma \in A$ in eJU_1 : $(\gamma.\gamma)[(\gamma.\kappa).(\gamma.\kappa)] \in A$. Using eJU_2 : $\gamma.\gamma = \gamma$ and $\gamma.\kappa = \kappa$ implies that: $\gamma.[(\gamma.\kappa).\kappa] \in A \Rightarrow (\gamma.\kappa).\kappa \leq \gamma \Rightarrow eJ_{10}$ is proved. □

Lemma 3.3. An eJU -algebra X satisfying, for any $\kappa, \gamma, z \in X$.

$(eJ_{11}) \kappa.\kappa \in A$,

$(eJ_{12}) [(z.\kappa).\kappa] \leq z$,

$(eJ_{13}) z.(\gamma.\kappa) = \gamma.(z.\kappa)$,

$(eJ_{14}) X$ is KU -algebra, if $(\kappa.\gamma).\gamma = 1$,

$(eJ_{15}) \kappa.\alpha_1 = \kappa.\alpha_2$ for all $\alpha_1, \alpha_2 \in A$,

$(eJ_{16}) (\gamma.\kappa).A = (\gamma.A).(\kappa.A)$.

Proof. The proof of eJ_{11} is similar to the proof of eJ_4 .

For eJ_{12} , take $\gamma = a \in A$ in eJU_1 , we get, $(a.z).[(z.\kappa).(a.\kappa)] \in A$.

By using eJU_2 , $z.[(z.\kappa).\kappa] \in A$. Implies,

$$[(z.\kappa).\kappa] \leq z. \tag{1}$$

As (eJ_7) is: $\kappa \leq \gamma$ implies $\gamma.z \leq \kappa.z$.

Let $z = \gamma.\kappa$ and using eJ_7 and (1):

$$(z.(\gamma.\kappa)) \leq ((z.\kappa).\kappa).(\gamma.\kappa). \quad (2)$$

Replacing z with $(z.\kappa)$ in eJU_1 : $(\gamma.(z.\kappa)).[((z.\kappa).\kappa).(\gamma.\kappa)] \in A$.

$$[((z.\kappa).\kappa).(\gamma.\kappa)] \leq (\gamma.(z.\kappa)). \quad (3)$$

As (eJ_6) is: $\kappa \leq z, z \leq \gamma$ imply $\kappa \leq \gamma$.

From (2), (3) and by using eJ_6 :

$$z.(\gamma.\kappa) \leq \gamma.(z.\kappa). \quad (4)$$

Replacing γ and z with z and γ respectively,

$$\gamma.(z.\kappa) \leq z.(\gamma.\kappa). \quad (5)$$

As (eJ_5) is: $\kappa \leq \gamma, \gamma \leq \kappa$ imply $\kappa = \gamma$.

From (4), (5) and by using eJ_5 : $z.(\gamma.\kappa) = \gamma.(z.\kappa) \Rightarrow eJ_{13}$ is proved.

Since every JU is an eJU with $A = \{1\}$.

To prove eJ_{13} we only have to show: $\kappa.1 = 1, \forall \kappa \in X$.

Let $z = \kappa, \gamma = 1$ and $\kappa = \gamma$ in eJU_1 , it gives $(1.\kappa).[(\kappa.\gamma).(1.\gamma)] = 1$.

$$\kappa.[(\kappa.\gamma).\gamma] = 1. \quad (6)$$

We have given that $(\kappa.\gamma).\gamma = 1$, putting in 6: $\kappa.1 = 1$, eJ_{14} is proved.

For any two elements $\alpha_1, \alpha_2 \in A$. The elements $\alpha_1.\alpha_2$ and $\alpha_2.\alpha_1 \in A$. By (eJU_1) and (eJU_2); $(\kappa.\alpha_1).(\kappa.\alpha_2) = (\kappa.\alpha_1).[(\alpha_1.\alpha_2).(\kappa.\alpha_2)] \in A$.

Similarly, $(\kappa.\alpha_2).(\kappa.\alpha_1) = (\kappa.\alpha_2).[(\alpha_2.\alpha_1).(\kappa.\alpha_1)] \in A$.

Now by eJ_5 ; $\kappa.\alpha_1 = \kappa.\alpha_2 \Rightarrow eJ_{15}$ is proved.

Taking any element $a \in A$.

$(\gamma.a).(\kappa.a) = (\gamma.a).[\kappa.((\gamma.\kappa).(\gamma.\kappa))]$ (by eJ_{11} and eJ_{15})

$= (\gamma.a).[(\gamma.\kappa).(\kappa.(\gamma.\kappa))]$ (by eJ_{13})

$= (\gamma.\kappa).[(\gamma.a).(\kappa.(\gamma.\kappa))]$

$= (\gamma.\kappa).[(\gamma.a).(\gamma.(\kappa.\kappa))]$

$= (\gamma.\kappa).[(\gamma.a).(\gamma.a)]$

$= (\gamma.\kappa).a$ (L.H.S), eJ_{16} is proved. □

Theorem 3.1. *If $(X, \cdot, 1)$ is a JU-algebra, then $(X, \cdot, \{1\})$ is an eJU-algebra but converse is not true.*

Remark 3.3. *Therefore eJU strictly generalizes JU; Example 3.2 shows the containment is proper.*

Example 3.1. *Consider $X = \{1, 2, 3\}$ is a set where $\cdot$ is described in the following table:*

We take $\gamma = 1, \kappa = 2, z = 3$.

$(JU_1) (\gamma.z).[(z.\kappa).(\gamma.\kappa)] = 1.$

By putting values, we have $= (1.3).[(3.2).(1.2)] = 3.(1.2) = 3.2 = 1.$

Hence (JU_1) is satisfied.

$\cdot$	1	2	3
1	1	2	3
2	1	1	2
3	1	1	1

Checking for (JU_2) :

$(JU_2): 1.\kappa = \kappa$

For set X we can see that: $1.1 = 1; 1.2 = 2; 1.3 = 3.$

Hence (JU_2) is satisfied.

Lets see (JU_3) :

$(JU_3): \kappa.\gamma = \gamma.\kappa = 1$ *implies $\kappa = \gamma$.*

For $\kappa = 1 = \gamma$, we have: $1.1 = 1.$

For $\kappa = 2 = \gamma$, we have: $2.2 = 1.$

For $\kappa = 3 = \gamma$, we have: $3.3 = 1.$

Hence (JU_3) is satisfied.

After checking X for all possible subsets, we have come to know that X is a JU-algebra. Hence every JU-algebra is an eJU-algebra if we take $A = \{1\}.$

Now we will show that not every eJU-algebra is a JU-algebra.

Example 3.2. *Consider $X = \{1, 2, 3\}$ is a set where $\cdot$ is described in the following table:*

Let $A = \{1, 2\}$

$\cdot$	1	2	3
1	1	2	3
2	1	2	3
3	1	2	1

We take $\gamma = 2, \kappa = 2, z = 3.$

Since $(eJU_1) (\gamma.z).[z.\kappa].(\gamma.\kappa) \in A$.

By putting values, we have: $= (2.3).[3.2].(2.2) = 3.(2.2) = 3.2 = 2 \in A$

Hence (eJU_1) is satisfied.

Checking for (eJU_2) ;

(eJU_2) : $A.\kappa = \{\kappa\}, \forall \kappa \in X$,

By putting values, we have

$1.2 = 2; 2.2 = 2$ which equals to κ .

Hence (eJU_2) is satisfied.

Checking for (eJU_3) ;

(eJU_3) : $\kappa.\gamma \in A, \gamma.\kappa \in A$ implies $\kappa = \gamma$,

For $\kappa = 2$ and $\gamma = 2$, we have $2.2 = 2 \in A$.

Hence (eJU_3) is satisfied.

After checking X for all the possible subsets, we have come to know that X is an eJU -algebra.

Now we will check if X is a JU -algebra or not?

We take $\gamma = 2, \kappa = 2, z = 3$.

$(JU_1) (\gamma.z).[z.\kappa].(\gamma.\kappa) = 1$.

By putting values, we have: $= (2.3).[3.2].(2.2) = 3.[2.2] = 3.2 = 2$ which does not equals to 1.

Hence proved that it is not a JU -algebra.

5. eJU -Ideals and minimals

Here we will discuss the concept of eJU -subalgebras, eJU -ideals and minimals with their related algebraic properties.

Definition 4.1. An eJU -subalgebra of X is defined as $J \neq \emptyset$ subset of set X satisfying $\gamma.\kappa \in J$, for all $\kappa, \gamma \in J$.

Definition 4.2. A p -semisimple eJU -algebra is an eJU -algebra X satisfying $(\kappa.A).A = \{\kappa\}$, for all $\kappa \in X$. A **minimal element** is an element j of X satisfying $\kappa \leq j$ implies $\kappa = j$, $\forall \kappa \in X$.

The branch of j for a minimal element $j \in X$, denoted as $K(j)$ and described as $K(j) := \{\kappa \in A \mid \kappa \geq j\}$.

The set $B_A = \{\kappa \in X \mid \kappa.A = A\}$ is called the eJU -part of X .

Definition 4.3. Suppose X is an eJU -algebra with a subset $J \neq \emptyset$ in it, is eJU -ideal if;

i) $A \subseteq J$,

ii) $\kappa.\gamma, \kappa \in J$ implies $\gamma \in J$, $\forall \kappa, \gamma \in X$.

Example 4.1. From eJU_1 , we can see that $(X, \cdot, 1)$ is an eJU -algebra.

Now let's see the eJU -ideals of X .

Let $R = \{1, 2\}$ be an ideal. Then $\kappa, \kappa.\gamma$ imply $\gamma \in J, \forall \kappa, \gamma \in X$.

Proof. $J = \{1, 2, 3\}$

$1.2 = 2 \in J, 2.1 = 1 \in J, 1.1 = 1 \in J, 2.2 = 2 \in J$.

Hence the given condition holds. □

Now let $S = \{1, 3\}$:

(i) $S \subseteq J$,

Proof. Obvious. □

(ii) $\kappa, \kappa.\gamma$ imply $\gamma \in J, \forall \kappa, \gamma \in X$.

Proof. $J = \{1, 2, 3\}$

$1.3 = 3 \in J, 3.1 = 1 \in J, 1.1 = 1 \in J, 3.3 = 1 \in J$.

Hence (ii) is satisfied. □

We can see that $R = \{1, 2\}$ and $S = \{1, 3\}$ are eJU -ideals of X .

Definition 4.4. A p -ideal of the set X is a subset J of an eJU algebra X satisfying: $A \subseteq J$ and $\gamma, (z.\gamma).(z.\kappa) \in J$ implies $\kappa \in J \forall \kappa, \gamma, z \in X$ and $A \subseteq X$.

Now we will discuss some additional lemmas associated with our definitions providing appropriate proofs to enhance the depth of our argument.

Lemma 4.1. Consider X is an eJU -algebra. Suppose $\kappa, \gamma \in X$. Then

i) $((\gamma.\kappa).\kappa)^n.\kappa = \gamma^n.\kappa$ for $n \in \mathbb{N}$.

ii) $(\kappa^n.A).A = (\kappa.A)^n.A$ for $n \in \mathbb{N}$.

Proof. i) Using induction method to prove these statements.

For $n = 0$, it's true.

Suppose it's true for $n = k$.

i.e $((\gamma.\kappa).\kappa)^k.\kappa = \gamma^k.\kappa$ for any $k \in \mathbb{N}$.

Now

$$\begin{aligned} & ((\gamma.\kappa).\kappa)^{k+1}.\kappa \\ &= (\gamma.\kappa).\kappa.((\gamma.\kappa).\kappa)^k.\kappa \\ &= (\gamma.\kappa).\kappa.\gamma^k.\kappa \\ &= \gamma^k.((\gamma.\kappa).\kappa.\kappa), \text{ by using } eJ_{12} \\ &= \gamma^k.(\gamma.\kappa), \text{ by using } eJ_{11} \end{aligned}$$

$$= \gamma^{k+1} \cdot \kappa.$$

So, this equality also applies to $n = k + 1$. $((\gamma \cdot \kappa) \cdot \kappa)^n \cdot \kappa = \gamma^n \cdot \kappa$ for $n \in \mathbb{N}$.

ii) Without any loss of generality, Suppose that $n \geq 1$. Now using the left distribution law on

$$(\kappa^n \cdot A) \cdot A = \kappa \cdot (\kappa^{n-1} \cdot A) \cdot A.$$

We get,

$$\begin{aligned} (\kappa^n \cdot A) \cdot A &= (\kappa \cdot A)(\kappa^{n-1} \cdot A) \cdot A \\ &= (\kappa \cdot A)^2 (\kappa^{n-2} \cdot A) \cdot A = (\kappa \cdot A)^3 (\kappa^{n-3} \cdot A) \cdot A = \dots = (\kappa \cdot A)^n \cdot A. \end{aligned}$$

□

Theorem 4.1. *The following properties are equivalent, consider X is an eJU-algebra and $j \in X$.*

i) j is minimal;

ii) $(j \cdot A) \cdot A = \{j\}; A \subseteq X$

iii) there is $\kappa \in X$ st $\{j\} = \kappa \cdot A; A \subseteq X$.

Proof. First prove (ii) by using (i),

By putting $\gamma \in A, z = j, \kappa \in A$ in eJU_1 ,

We get; $(A \cdot j) \cdot [(j \cdot A) \cdot (A \cdot A)] \subseteq A$

$j \cdot [(j \cdot A) \cdot A] \subseteq A$, by using eJU_2 .

Since j is minimal if $\kappa \leq j$ implies $\kappa = j$

Here $\{\kappa\} = (j \cdot A) \cdot A$.

$\forall a \in A, (j \cdot a) \cdot a \leq j$ implies $(j \cdot a) \cdot a = j$.

Now proving (i) by using (iii),

Given that: $\{j\} = \kappa \cdot A$ for an $\kappa \in X$.

For $\gamma \in X$, if $j \cdot \gamma \in A$, then $(\kappa \cdot a) \cdot \gamma \in A$.

For all $a \in A$, we have;

$$\begin{aligned} \gamma \cdot j &= \gamma \cdot (\kappa \cdot a) \\ &= \gamma \cdot (((\kappa \cdot a) \cdot a) \cdot a), \text{ using } eJ_{10} \\ &= ((\kappa \cdot a) \cdot a) \cdot (\gamma \cdot a), \text{ using } eJ_{12} \\ &= ((\kappa \cdot a) \cdot \gamma) \cdot a, \\ &= a \cdot a = a, \text{ since } (\kappa \cdot a) \cdot \gamma \in A. \end{aligned}$$

Thus $\gamma \cdot j \in A$.

Additionally, $j.\gamma \in A$. Using eJU_3 , we get $\gamma = j$. Hence j is a minimal element of X . Now proving (iii) using (ii)., By hypothesis, $\{j\} = (j.A).A$. Assume that $j.A = \{\kappa\}$ for some $\kappa \in X, A \subseteq X \Rightarrow \{j\} = \kappa.A$ for some $\kappa \in X$, here $\{\kappa\} = j.A$. Hence $\{j\} = \kappa.A$. □

5. Conclusion

In this work, we defined the extension of JU algebras where we opted for a non-empty subset in place of special element 1. We established a relation $\leq$ within the set X and also specified that for any element κ in the set A , we have $A.\kappa = \{\kappa\}$. We presented a lemma concerning partially ordered sets (POSET) in eJU algebras, outlining certain properties, and demonstrate the validity of these properties with the help of axioms of eJU algebras. We state certain inequalities and specify the condition under which X qualifies as a KU algebra. We also study the conditions where the JU algebra is an eJU algebra but the converse does not exist. We introduced p-semisimple eJU algebras, eJU-ideals, eJU-subalgebras, minimals and p-ideals in eJU algebras defining their algebraic properties. This extension represents a new approach to the previously established concept of JU-algebras.

Acknowledgement

The authors gratefully acknowledge the valuable comments and constructive suggestions of the anonymous referee, which have greatly improved the quality of this paper.

References

- [1] M. Panteki, "The mathematical background of George Boole's mathematical analysis of logic," A Boole Anthology: Recent and Classical Studies in the Logic of George Boole, vol. 1847, pp. 167–212, 2000.
- [2] Y. Imai and I. K, "On axiom systems of propositional calculi," Proceedings of the Japan Academy, vol. 42, pp. 19–22, 1966.
- [3] I. K, "An algebra related with a propositional calculus," Proceedings of the Japan Academy, vol. 42, pp. 26–29, 1966.
- [4] Q. P. Hu and X. Li, "On BCH-algebras," Math. Sem. Notes Kobe Univ., vol. 11, pp. 313–320, 1983.
- [5] Y. Komori, "The class of bcc-algebras is not a variety," Math. Japonica, vol. 29, pp. 391–394, 1984.

- [6] W. A. Dudek, "On proper bcc-algebras," *Bulletin of the Institute of Mathematics Academic Science*, vol. 20, pp. 137–150, 1992.
- [7] Y. B. Jun, E. H. Roh, and H. S. Kim, "On BH algebras," *International society of Mathematical sciences*, pp. 347–354, 1998.
- [8] M. A. Chaudhry, A. Fahad, Y. Rao, M. I. Qureshi, and S. Gulzar, "Branchwise solid generalized bch-algebras," *AIMS Mathematics*, vol. 5, no. 3, pp. 2424–2432, 2020.
- [9] M. A. Chaudhry, M. I. Qureshi, A. Fahad, and M. S. Bashir, "Isomorphism theorems in generalized d-algebras," *Journal of Prime Research in Mathematics*, vol. 17, no. 2, pp. 149–158, 2022.
- [10] M. A. Chaudhry, A. Fahad, M. I. Qureshi, and U. Riasat, "Some results about weak UP-algebras," *Journal of Mathematics*, vol. 2022, pp. 1–6, 2022.
- [11] W. Rump, "L-algebras and three main non-classical logics," *Annals of Pure and Applied Logic*, vol. 173, no. 7, p. 103121, 2022.
- [12] A. Radfar, A. Rezaei, A. Saeid, and L. Liu, "Extensions of bck-algebras," *Cogent Mathematics*, vol. 3, no. 1, p. 1265297, 2016.
- [13] C. Prabpayak and U. Leerawat, "On ideals and congruences in ku-algebras," *Scientia Magna*, vol. 5, no. 1, pp. 54–57, 2009.
- [14] M. A. Ansari, A. Haider, and A. N. Koam, "On ju-algebras and p-closure ideals," *International Journal of Mathematics and Computer Science*, vol. 15, no. 1, pp. 135–154, 2020.
- [15] D. A. Romano, "A few comments and some new results on ju-algebras," *Open Journal of Mathematical Sciences*, vol. 4, no. 1, pp. 110–117, 2020.
- [16] M. A. Ansari, "Rough set theory applied to JU-algebras," *International Journal of Mathematics and Computer Science*, vol. 16, pp. 1371–1384, 2021.
- [17] A. Haider, "On a soft quotient structure over JU-algebras," *Int. J. Anal. Appl.*, vol. 23, p. 56, 2025.
- [18] A. H. Hakami, M. A. Ansari, and A. Haider, "On some graphs based on the ideals of JU-algebras," *Int. J. Anal. Appl.*, vol. 22, p. 1, 2024.
- [19] A. Satirad, R. Chinram, and A. Iampan, "Four new concepts of extensions of KU/UP-algebras," *Missouri Journal of Mathematical Sciences*, vol. 32, no. 2, pp. 138–157, 2020.